SOLUTION – BANK

PEYAM RYAN TABRIZIAN

This document contains solutions to various Math 1A homework problems I've compiled, from Fall 2010, Spring 2011, and Fall 2013. At first, the solutions might seem disappointingly short, but don't worry! As the course progresses, the solutions become longer and more detailed! It contains solutions to the following problems from the 7th edition of the book (see next page):

Date: Friday, December 13, 2013.

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3.4	12, 15, 39, 45, 49, 63, 66, 67, 68, 72, 80, 85, 86, 92
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5.3	7, 15, 17, 31, 35, 37, 39, 43, 45, 57, 67, 70
5.4	10, 12, 13, 25, 31, 37, 49, 54, 61, 62, 63, 64
5.5	7, 31, 33, 48, 59, 62, 77, 86, 92
6.1	1, 3, 13, 21, 42, 43, 51
6.2	6, 13, 17, 47, 49, 55, 65, 68
6.3	2, 13, 15, 19, 46, 48

Section 1.1: Four ways to represent functions

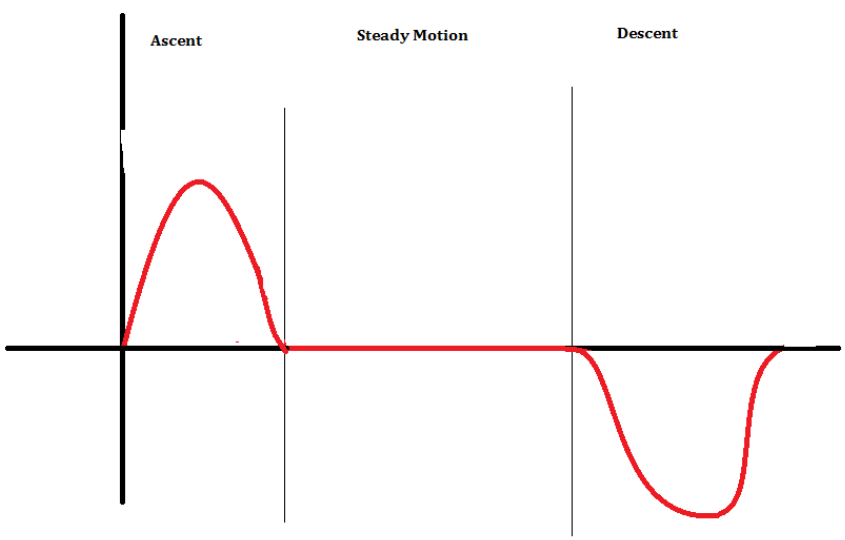
1.1.7. No (by the vertical line test)

1.1.8. Yes (by the vertical line test), Domain $= [-2, 2]$, Range $= [-1, 2]$

1.1.22.

(a) The graph of $x(t)$ should just be a line going through the origin
(b) The graph of $y(t)$ should look at first like the right half of a parabola, then should be constant for a while, and then look like the left half of a parabola
(c) The graph of the horizontal velocity looks like a horizontal line
(d) See announcement on bspace for a detailed solution! The picture you get is:

1A/Math 1A Summer/Solution Bank/Vertical Velocity.png



1.1.38. Domain $= [-2, 2]$, Range $= [0, 2]$, Graph is just the upper-half of the circle centered at 0 of radius 2.

1.1.51. $f(x) = \frac{5}{2}x - \frac{11}{2}$

1.1.63. $V(x) = x(20 - 2x)(12 - 2x)$ (no need to expand the answer!)

1.1.69. f is odd, g is even

Section 1.2: Mathematical models: a catalog of essential functions

1.2.2.
(a) Exponential function
(b) Power function
(c) Polynomial of degree 5
(d) Trigonometric function
(e) Rational function
(f) Algebraic function

1.2.4.
(a) G
(b) f
(c) F
(d) g

1.2.8. (a) $y = 2(x-3)^2$, (b) $y = -x^2 - \frac{5}{2}x + 1$

1.2.16.
(a) $C(x) = 13x + 900$ (C is the cost and x is the number of chairs produced)
(b) 13; Cost per chair
(c) 900; Start-up cost (i.e. money needed to buy machines in order to *start* producing chairs)

Section 1.3: New functions from old functions

1.3.1.
(a) $y = f(x) + 3$
(b) $y = f(x) - 3$
(c) $y = f(x-3)$
(d) $y = f(x+3)$
(e) $y = -f(x)$
(f) $y = f(-x)$
(g) $y = 3f(x)$
(h) $y = \frac{1}{3}f(x)$

1.3.7. $y = -\sqrt{3(x+4) - (x+4)^2} - 1$

1.3.14. Basically compress the graph of $\sin(x)$ horizontally by a factor of 3 (notice that the new period now is $\frac{2\pi}{3}$ and then stretch the resulting graph vertically by a factor of 4 (so the new graph has range $[-4, 4]$ instead of $[-1, 1]$)

1.3.30.
(a) $(f+g)(x) = \sqrt{3-x} + \sqrt{x^2-1}$
(b) $(f-g)(x) = \sqrt{3-x} + \sqrt{x^2-1}$
(c) $(fg)(x) = \sqrt{3-x} \times \sqrt{x^2-1}$
(d) $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{3-x}}{\sqrt{x^2-1}}$

All of those functions have domain $(-\infty, -1] \cup [1, 3]$ **EXCEPT** for (d), which has domain $(-\infty, -1) \cup (1, 3]$

1.3.36.

(a) $(f \circ g)(x) = \frac{\sin(2x)}{1+\sin(2x)}$; Dom = all odd multiples of $\frac{\pi}{2}$
(b) $(g \circ f)(x) = \sin\left(\frac{2x}{1+x}\right)$; Dom = all real numbers except -1
(c) $(f \circ f)(x) = \frac{\frac{x}{1+x}}{1+\frac{x}{1+x}} = \frac{x}{1+2x}$; Dom = all real numbers except $\frac{-1}{2}$ and -1
(d) $(g \circ g)(x) = \sin(2\sin(2x))$; Dom = all real numbers

Section 1.5: Exponential Functions

1.5.2. (a) 16; (b) $27x^7$

1.5.4. (a) x^{4n-3}; (b) $a^{\frac{1}{6}}b^{-\frac{1}{12}}$

1.5.7. Basically, the larger the base, the faster the function is increasing

1.5.9. Notice that $\left(\frac{1}{3}\right)^x = 3^{-x}$, which means that $\left(\frac{1}{3}\right)^x$ is the reflection of 3^x across the y-axis! Similarly with 10^x.

1.5.17.

(a) $y = e^x - 2$
(b) $y = e^{x-2}$
(c) $y = e^{-x}$
(d) $y = -e^x$
(e) $y = -e^{-x}$

1.5.20. (a) All real numbers ; (b) All ≤ 0 real numbers

1.5.21. $f(x) = 3 \cdot 2^x$

Section 1.6: Inverse functions and logarithms

1.6.3. No; For example, even though $2 \neq 6$, $f(2) = f(6) = 2$

1.6.5. No (by the horizontal line test)

1.6.6. Yes (by the horizontal line test)

1.6.17. 0 (You want to find x such that $g(x) = 4$, that is, find x such that $x+e^x = 1$. Here, just guess!)

1.6.18.

(a) By the horizontal line test
(b) Domain of f^{-1} = Range of $f = [-1, 3]$; Range of f^{-1} = Domain of $f = [-3, 3]$
(c) 0
(d) ≈ -1.8

1.6.26. $f^{-1}(x) = \ln\left(-\frac{x}{2x-1}\right) = \ln\left(\frac{x}{1-2x}\right)$

1.6.52.

(a) $x = \pm\sqrt{1+e^3}$
(b) $x = 0, \ln(2)$ (Let $X = e^x$ and solve the equation $X^2 - 3X + 2 = 0$ (by using the quadratic formula), then solve for x using $e^x = X$)

1.6.63.

(a) $\frac{\pi}{3}$
(b) π

1.6.69. If $\theta = \sin^{-1}(x)$, then $\sin(\theta) = x$, then draw a triangle with hypothenuse 1, and opposite side x, and then the adjacent side becomes $\sqrt{1-x^2}$, and so our answer becomes:

$$\cos(\sin^{-1}(x)) = \cos(\theta) = \frac{adjacent}{hypotenuse} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

See the handout "Proof of the derivative of arccos" for a similar problem; Or look at your notes taken in section!

1.6.70. $\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1}(x))}{\cos(\sin^{-1}(x))} = \frac{x}{\sqrt{1-x^2}}$ by the result of number 69!

Section 2.2: The limit of a function

2.2.2. If x approaches 1 from the left, then $f(x)$ approaches 3; If x approaches 1 from the right, then $f(x)$ approaches 7. No, left-hand-limits and right-hand-limits must be equal!

2.2.6.

(a) 4
(b) 4
(c) 4
(d) Undefined
(e) 1
(f) -1
(g) Does not exist (left and right-side limits not equal)
(h) 1
(i) 2
(j) Undefined
(k) 3
(l) Does not exist (h does not approach one fixed value as x approaches 5 from the left)

2.2.32. $-\infty$ (numerator approaches $e^{-5} > 0$ while denominator approaches 0^-

2.2.33. $-\infty$ ($x^2 - 9$ approaches 0^+ and $\ln(0^+) = -\infty$

2.2.46. The mass blows up to ∞ ($\frac{v^2}{c^2}$ goes to 1^-, so the denominator of the fraction goes to 0^+, and so the whole fraction goes to ∞)

Section 2.3: Calculating limits using the limit laws

2.3.9. Just plug in $x = 2$

2.3.10.

(a) If you plug in $x = 2$, then the left hand side is not defined, but the right hand side is

(b) The above equation holds if $x \neq 2$, but the point of limits is that in this case you don't **care** about the value at 2! So in this case, the equality is correct!

2.3.13. Does not exist (left-hand-limit is $-\infty$ because the numerator tends to 4 and the denominator tends to 0^- while the right-hand-limit is ∞ because the numerator tends to 4 and the denominator tends to 0^+)

2.3.18. 12 (Use the formula $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$)

2.3.20. 3 (use the fact that $x^3 - 1 = (x - 1)(x^2 + x + 1)$)

2.3.26. 1 (put under a common denominator $t^2 + t = t(t + 1)$ and cancel out)

2.3.29. $\frac{1}{2}$ (put under a common denominator and multiply by the conjugate form)

2.3.32. $-\frac{2}{x^3}$ (put under a common denominator and expand the numerator out)

2.3.37. 7 (use the squeeze theorem)

2.3.40. 0 (by squeeze theorem, because $-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$)

2.3.60. Let $a = 0$ and $f(x) = \sin\left(\frac{1}{x}\right)$ (or $\frac{1}{x}$), and $g(x) = -f(x)$.

2.3.61. Let $a = 0$ and $f(x) = \sin\left(\frac{1}{x}\right)$ (or $\frac{1}{x}$), and $g(x) = \frac{1}{f(x)}$

2.3.64. Hints: Use the following steps:

(a) Find the coordinates of Q. For this, solve for x and y in the system of equations:

$$\begin{cases} (x-1)^2 + y^2 = 1 \\ x^2 + y^2 = r^2 \end{cases}$$

For this, plug in $y^2 = r^2 - x^2$ in the first equation and solve for x, then solve for y in $y^2 = r^2 - x^2$; remember that you want $x > 0$ and $y > 0$, according to the picture). The answer gives you the coordinates of Q

(b) Now that you know the coordinates of P and Q, find the equation of the line going through P and Q

(c) Find the x-intercept of that line (set $y = 0$ and solve for x)

(d) Finally, take the limit as $r \to 0^+$ of the answer you found in (c). To do this, multiply as usual by the conjugate form.

Answers:

(a) $Q = \left(\frac{r^2}{2}, r\sqrt{1 - \frac{r^2}{4}}\right)$

(b) $y = \frac{2}{r}\left(\sqrt{1 - \frac{r^2}{4}} - 1\right)x + r$

(c) $x-$ intercept $= \frac{r^2}{2\left(1 - \sqrt{1 - \frac{r^2}{4}}\right)}$

(d) $\boxed{4}$

Section 2.4: The precise definition of a limit

2.4.2. $\delta = 0.4$

2.4.4. $\delta = min\left\{1, \frac{1}{6}\right\} = \frac{1}{6}$ (there are many answers to this question, so don't worry if yours is different from mine)

2.4.11. Note: If this problem is too confusing for you, skip it! (it does more harm than good)

(a) $r = \sqrt{\frac{1000}{\pi}}$ (from now on, let's call this a)
(b) $|r-a| < min\left\{\frac{5}{\pi(2a+1)}, 1\right\} = \frac{5}{\pi(2a+1)}$ (for this, first set $|r-a| < 1$, solve for r to get $a-1 < r < a+1$, then $|r+a| = r+a < 2a+1$, so $\pi|r-a||r+a| < \pi|r-a|(2a+1) < 5$, hence $|r-a| < \frac{5}{\pi(2a+1)}$
(c) $x = r =$ radius, $a = \sqrt{\frac{1000}{\pi}}$, $L = 1000$, $\epsilon = 5$, $\delta = min\left\{1, \frac{5}{\pi(2a+1)}\right\}$

2.4.19. This is an example of the 'easy case' with $\delta = \frac{3\epsilon}{4}$

2.4.32. This is an example of the 'complicated case' with $\delta = \min\left\{1, \frac{\epsilon}{19}\right\}$.

To get this δ, notice that if $|x-2| < 1$, then $1 < x < 3$, and so $7 < x^2+2x+4 < 19$, so $|x^2+2x+4| < 19$

2.4.37. This is again an example of the 'complicated case' with $\delta = min\left\{\frac{a}{2}, \epsilon\sqrt{a}\left(1+\frac{1}{\sqrt{2}}\right)\right\}$

To get this δ, notice that if $|x-a| < \frac{a}{2}$, then $\frac{a}{2} < x < \frac{3a}{2}$, and so in particular $\sqrt{x}+\sqrt{a} > \left(1+\frac{1}{\sqrt{2}}\right)\sqrt{a}$ and then:

$$\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < |x-a|\frac{1}{\left(1+\frac{1}{\sqrt{2}}\right)\sqrt{a}} < \epsilon$$

which gives:

$$|x-a| < \epsilon\sqrt{a}\left(1+\frac{1}{\sqrt{2}}\right)$$

The next two are optional, but good for practice:

2.4.42. $\delta = \sqrt[4]{\frac{1}{M}}$

2.4.43. $\delta = e^M$ (where M is negative)

2.4.44. This is absolutely ridiculous, so feel free to skip it if you want!

(a) Let $M > 0$. We want to find δ such that if $|x-a| < \delta$, then $f(x)+g(x) > M$.

However, by replacing M by $M+(1-c)$ in the definition of the limit of f, there exists δ_1 such that if $|x-a| < \delta_1$, then $f(x) > M+(1-c)$

Moreover, by letting $\epsilon = 1$ in the definition of the limit of g, we know that there exits δ_2 such that if $|x-a| < \delta_2$, then $|g(x)-c| < 1$. In particular, we get $g(x)-c > -1$, so $g(x) > c-1$.

Hence, if you choose $\delta = min\{\delta_1, \delta_2\}$, then if $|x-a| < \delta$, we have $f(x)+g(x) > M+(1-c)+(c-1) = M$.

(b) Let $M > 0$. We want to find $\delta > 0$ such that if $|x-a| < \delta$, then $f(x)g(x) > M$.

By replacing M by $M\left(\frac{2}{c}\right)$ in the definition of the limit of f, we know that there exists δ_1 such that if $|x-a| < \delta_1$, then $f(x) > M\left(\frac{2}{c}\right)$

Moreover, by letting $\epsilon = \frac{c}{2} > 0$ in the definition of the limit of g, we know that there exists δ_2 such that if $|x-a| < \delta_2$, then $|g(x)-c| < \frac{c}{2}$. In particular, $g(x)-c > -\frac{c}{2}$, so $g(x) > c-\frac{c}{2} = \frac{c}{2} > 0$.

Hence, if you choose $\delta = min\{\delta_1, \delta_2\}$, then if $|x-a| < \delta$, we have $f(x)g(x) > \left(M\frac{2}{c}\right)\left(\frac{c}{2}\right) = M$.

(c) Let $\boxed{M < 0}$. We want to find $\delta > 0$ such that if $|x-a| < \delta$, then $f(x)g(x) < M$.

By replacing M by $M\left(\frac{2}{c}\right)$ in the definition of the limit of f, we know that there exists δ_1 such that if $|x-a| < \delta_1$, then $f(x) > M\left(\frac{2}{c}\right)$

Moreover, by letting $\epsilon = -\frac{c}{2} > 0$ in the definition of the limit of g, we know that there exists δ_2 such that if $|x-a| < \delta_2$, then $|g(x)-c| < -\frac{c}{2}$. In particular, $g(x)-c < -\frac{c}{2}$, so $g(x) < c-\frac{c}{2} = \frac{c}{2} < 0$.

Hence, if you choose $\delta = min\{\delta_1, \delta_2\}$, then if $|x-a| < \delta$, we have $f(x)g(x) < \left(M\frac{2}{c}\right)\left(\frac{c}{2}\right) = M$. (Note that in fact the $>$ in the first part, becomes a $<$ here precisely because $g(x) < 0$!)

Section 2.5: Continuity

2.5.3. -4 (f not defined at -4; neither), -2 (left-hand-side and right-hand-side limits not equal; continuous from the left), 2 (ditto; continuous from the right), 4 (left-hand-side limit does not exist; continuous from the right)

2.5.11. $f(2) = 4$ (You get this by solving for $f(2)$ in $3f(2)+f(2)g(2) = 36$)

2.5.20. Not continuous because the limit as $x \to 1$ equals $\frac{1}{2}$ (factor out the numerator and denominator), whereas $f(1) = 1$

2.5.28. Continuous because it's a ratio of two continuous functions (and the numerator/denominator are continuous because of composition of continuous functions), $Dom = \mathbb{R}$

2.5.38. $\tan^{-1}\left(\frac{2}{3}\right)$

2.5.44. Yes, you can check that the left-hand-side-limits and the right-hand-side limits are equal! Plug in values for G, M, R if you want to, for example $G = 2$, $M = 5$, $R = 7$

2.5.51. Define $f(x) = x^4 + x - 3$, then $f(1) = -1 < 0$, $f(2) = 15 > 0$, so by IVT, there is one number c such that $f(c) = 0$.

2.5.54. Let $f(x) = \sin(x) - x^2 + x$

Then $f(1) = \sin(1) - 1 + 1 = \sin(1) > 0$, whereas $f(2) = \sin(2) - 4 + 2 = \sin(2) - 2 < 0$.

Moreover, f is **continuous** on $[1, 2]$, hence by the **Intermediate Value Theorem** there exists one c in $(1, 2)$ such that $f(c) = 0$, that is, $\sin(c) = c^2 - c$

2.5.60. Use the fact that $\sin(a + h) = \sin(a)\cos(h) + \sin(h)\cos(a)$

2.5.69. Define $f(t)$ to be the altitude of the monk on the first day, $g(t)$ to be the altitude of the monk on the second day, and let $h(t) = f(t) - g(t)$. Then $h(0) > 0$, $h(12) < 0$ (where 0 means $7AM$ and 12 means $12PM$), then by IVT, there is one number c such that $h(c) = 0$, i.e. $f(c) = g(c)$

Section 2.6: Limits at Infinity; Horizontal Asymptotes

2.6.4.

(a) 2
(b) -1
(c) $-\infty$
(d) $-\infty$
(e) ∞
(f) Horizontal asymptotes: $y = -1$, $y = 2$; Vertical asymptotes: $x = 0$, $x = 2$

2.6.16. 0 (factor out x^3 from the numerator and the denominator)

2.6.26.

$$\begin{aligned}
\lim_{x\to-\infty} x+\sqrt{x^2+2x} &= \lim_{x\to-\infty} x+\sqrt{x^2}\sqrt{1+\frac{2}{x}} \\
&= \lim_{x\to-\infty} x+|x|\sqrt{1+\frac{2}{x}} \\
&= \lim_{x\to-\infty} x-x\sqrt{1+\frac{2}{x}} \\
&= \lim_{x\to-\infty} x\left(1-\sqrt{1+\frac{2}{x}}\right) \\
&= \lim_{x\to-\infty} x\left(1-\sqrt{1+\frac{2}{x}}\right)\frac{\left(1+\sqrt{1+\frac{2}{x}}\right)}{\left(1+\sqrt{1+\frac{2}{x}}\right)} \\
&= \lim_{x\to-\infty} x\left(\frac{1^2-\left(\sqrt{1+\frac{2}{x}}\right)^2}{1+\sqrt{1+\frac{2}{x}}}\right) \\
&= \lim_{x\to-\infty} x\left(\frac{1-\left(1+\frac{2}{x}\right)}{1+\sqrt{1+\frac{2}{x}}}\right) \\
&= \lim_{x\to-\infty} x\left(\frac{1-1-\frac{2}{x}}{1+\sqrt{1+\frac{2}{x}}}\right) \\
&= \lim_{x\to-\infty} x\left(\frac{-\frac{2}{x}}{1+\sqrt{1+\frac{2}{x}}}\right) \\
&= \lim_{x\to-\infty} \frac{-2}{1+\sqrt{1+\frac{2}{x}}} \\
&= \frac{-2}{1+1} \\
&= -1
\end{aligned}$$

2.6.38. $\tan^{-1}(-\infty) = -\frac{\pi}{2}$ (by continuity of $\tan^{-1}$)

2.6.43. $y = 2$ (at $\pm\infty$), $x = 1$, $x = -2$ (factor out the denominator)

2.6.61. 5 (by the squeeze theorem)

Section 2.7: Derivatives and Rates of Change

2.7.6. $y = 9x - 15$

2.7.12.

(a) A runs with constant speed, while B is slow at first and then speeds up
(b) ≈ 8.5 seconds
(c) 9 seconds

2.7.17. $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$

2.7.18. $y = 4x - 23$ ($y + 3 = 4(x - 5)$ is also acceptable)

2.7.19. $f(2) = 3$, $f'(2) = 4$

2.7.32. $f(x) = \sqrt[4]{x}$, $a = 16$

2.7.36. $f(x) = \tan(x)$, $a = \frac{\pi}{4}$

2.7.40. $f'(t) = -\frac{1}{t^2} - 1$ (show this, using the **definition** of the derivative) Velocity $= f'(5) = -\frac{26}{25}$ meters per second, Speed $= \frac{26}{25}$ meters/second

2.7.42. $\approx -\frac{5}{6}$ F/min (slope of the red line)

2.7.48.

(a) Rate of bacterias/hour after 5 houts
(b) $f'(10) > f'(5)$ (basically, the more bacteria there are, the more can be produced). But if there's a limited supply of food, we get that $f'(10) < f'(5)$, i.e. bacterias are dying out because of the limited supply

Section 2.8: The derivative as a function

2.8.3.

(a) II
(b) IV
(c) I
(d) III

2.8.23. $f'(t) = 5 - 18t$

2.8.40. -1 (not continuous there); 2 (graph has a kink)

2.8.43.

(a) Acceleration
(b) Velocity
(c) Position

2.8.54. Not differentiable at the integers, because not continuous there; $f'(x) = 0$ for x not an integer, undefined otherwise. Graph looks like the 0-function, except it has holes at the integers.

Section 3.1: Derivatives of polynomials and exponential functions

3.1.20. $S'(R) = 8\pi R$

3.1.32. $y' = e^{x+1}$

3.1.35. $y' = 4x^3 + 2e^x$, so $y'(0) = 2$, and so the equation of tangent line is $y - 2 = 2(x - 0)$, i.e. $\boxed{y = 2x + 2}$ and equation of normal line is $y - 2 = -\frac{1}{2}(x - 0)$, i.e. $\boxed{y = -\frac{1}{2}x + 2}$ (remember that the normal line still goes through $(0, 2)$, but has slope = the negative reciprocal of the slope of the tangent line)

3.1.47.
(a) $v(t) = s'(t) = 3t^2 - 3$; $a(t) = v'(t) = 6t$
(b) $a(2) = 12$
(c) $v(t) = 0$ if $t = 1$ or $t = -1$, but $t > 0$ (negative time doesn't make sense), so $t = 1$, and $a(1) = 6$

3.1.54. First of all $y' = \frac{3}{2}\sqrt{x}$ and first find a point x where $y'(x) = 3$ (remember that two lines are parallel when their slopes are equal, and the slope of $y = 1+3x$ is 3). So you want $\frac{3}{2}\sqrt{x} = 3$, so $\sqrt{x} = 2$, so $x = 4$. Now all that you need to find out is the slope of the tangent line to the curve at 4. The equation is: $\boxed{y - 8 = 3(x-4)}$ (because from the above calculation the slope is 3, and the tangent line goes through $(4, f(4)) = (4, 8)$)

Section 3.2: The product and quotient rules

3.2.15. $y' = \frac{2t\left(t^4-3t^2+1\right)-\left(t^2+2\right)\left(4t^3-6t\right)}{\left(t^4-3t^2+1\right)^2}$

3.2.33. $y'(x) = 2e^x + 2xe^x$, so $y'(0) = 2$, and so the tangent line has equation: $y - 0 = 2(x-0)$,i.e $\boxed{y = 2x}$, and the normal line has equation: $y - 0 = -\frac{1}{2}(x-0)$, i.e. $\boxed{y = -\frac{1}{2}x}$

3.2.41. $f'(x) = \frac{2x(1+x)-x^2}{(1+x)^2} = \frac{x^2+2x}{x^2+2x+1}$, so $f''(x) = \frac{(2x+2)(x^2+2x+1)-(x^2+2x)(2x+2)}{(x^2+2x+1)^2}$, and so $f''(1) = \frac{(2+2)(1+2+1)-(1+2)(2+2)}{(1+2+1)^2} = \frac{(4)(4)-(3)(4)}{(4)(4)} = \frac{16-12}{16} = \frac{4}{16} = \boxed{= \frac{1}{4}}$

3.2.49.
(a) $u'(1) = f'(1)g(1) + f(1)g'(1) = (2)(1) + (2)(-1) = \boxed{0}$
(b) $v'(5) = \frac{f'(5)g(5)-f(5)g'(5)}{(g(5))^2} = \frac{\left(-\frac{1}{3}\right)(2)-(3)\left(\frac{2}{3}\right)}{4} = \frac{-\frac{2}{3}-2}{4} = -\frac{8}{12} = \boxed{-\frac{2}{3}}$

3.2.57. (9200)(30593) + (961400)(1400) = 1,627,000

Section 3.3: Derivatives of trigonometric functions

3.3.37. We have $\sin(\theta) = \frac{x}{10}$, so $x = 10\sin(\theta)$, so $x'(\theta) = 10\cos(\theta)$, and $x'\left(\frac{\pi}{3}\right) = 10\cos\left(\frac{\pi}{3}\right) = \frac{10}{2} = \boxed{5}$

3.3.39. 3 (multiply the fraction by $\frac{3}{3}$ and use the fact that $\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1$)

3.3.40. $\frac{4}{6} = \frac{2}{3}$ (multiply the numerator by $\frac{4}{4}$ and the denominator by $\frac{6}{6}$ and use the facts that $\lim_{x\to 0} \frac{\sin(4x)}{4x} = 1$ and $\lim_{x\to 0} \frac{\sin(6x)}{6x} = 1$)

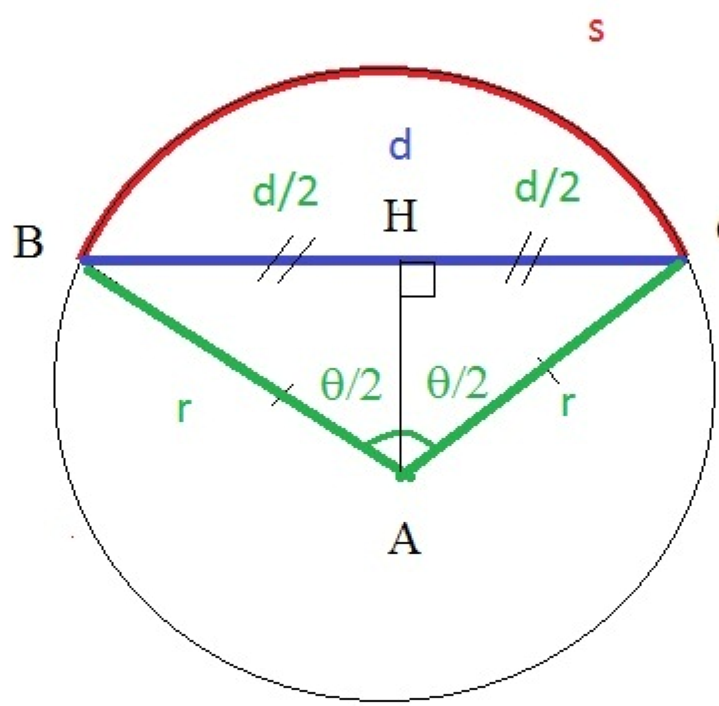
3.3.55. The problem is to calculate:

$$\lim_{\theta\to 0^+} = \frac{s}{d}$$

First of all, let's call the radius of the circle r.

Now, let's divide this into three simple sub-problems:

1A/Math 1A Summer/Solution Bank/Arclength.png



(1) **Calculate s**

But this is not very hard! Either you know about the formula for the length of an arc, or you can easily derive it! Namely, if an angle of 2π radians corresponds to a length of $2\pi r$ (i.e. the circumference of a circle), then an angle of θ radians corresponds to a length of s.
Now, because the length of an arc is proportional to the angle, we have the following equality:

$$\frac{s}{\theta} = \frac{2\pi r}{2\pi} = r$$

So $\boxed{s = \theta r}$

Notice that the use of radians makes this calculation particularly simple!

(2) **Calculate d**

This is a bit harder than the above step, but actually not that bad! Label the triangle in the figure ABC, and let H be the midpoint of BC.

Then, the triangle AHB is right in A, and we can use our regular definition of sin to find a relationship between r and d, namely:

$$sin(\angle BAH) = \frac{BH}{AB}$$

But you can easily check that $\angle BAH = \frac{\theta}{2}$, that $BH = \frac{d}{2}$ (because H is the midpoint of BC), and that $AB = r$! Hence, we get:

$$\sin(\frac{\theta}{2}) = \frac{\frac{d}{2}}{r}$$

That is, $\boxed{d = 2rsin(\frac{\theta}{2})}$.

(3) Compute the limit

The last thing we need to do is to calculate the required limit! But this is easy, since we know the values of s and d in terms of r:

$$\lim_{\theta\to 0^+} \frac{s}{d} = \lim_{\theta\to 0^+} \frac{r\theta}{2r\sin(\frac{\theta}{2})} = \lim_{\theta\to 0^+} \frac{\theta}{2\sin(\frac{\theta}{2})} = \lim_{\theta\to 0^+} \frac{\frac{\theta}{2}}{\sin(\frac{\theta}{2})}$$

Finally, let $t = \frac{\theta}{2}$ and notice that, as $\theta \to 0^+$, $t \to 0^+$, then, our limit becomes:

$$\lim_{\theta\to 0^+} \frac{s}{d} = \lim_{t\to 0^+} \frac{t}{\sin(t)} = \lim_{t\to 0^+} \frac{1}{\frac{\sin(t)}{t}} = \frac{\lim_{t\to 0^+} 1}{\lim_{t\to 0^+} \frac{\sin(t)}{t}} = \frac{1}{1} = 1$$

And again, the next-to-last step is justified because both limits (numerator and denominator) exist and the limit of the denominator is nonzero!

Hence, we get:

$\boxed{\lim_{\theta\to 0^+} \frac{s}{d} = 1}$

Section 3.4: The Chain Rule

3.4.12. $f'(t) = \cos(e^t)\, e^t + e^{\sin(t)} \cos(t)$

3.4.15. $y' = e^{-kx} - kxe^{-kx}$

3.4.39. $f'(t) = \sec^2(e^t) + e^{\tan(t)} \sec^2(t)$

3.4.45. $y' = -\sin(\sqrt{\sin(\tan(\pi x))})\frac{1}{2\sqrt{\sin(\tan(\pi x))}}\cos(\tan(\pi x))\sec^2(\pi x)\pi$

3.4.49. $y' = \alpha e^{\alpha x}\sin(\beta x) + e^{\alpha x}\beta\cos(\beta x)$; $y'' = e^{\alpha x}((\alpha^2 - \beta^2)\sin(\beta x) + 2\alpha\beta\cos(\beta x))$

3.4.63.

(a) $h'(1) = f'(g(1))g'(1) = f'(2)g'(1) = 5 \times 6 = 30$
(b) $H'(1) = g'(f(1))f'(1) = g'(3)f'(1) = 9 \times 4 = 36$

3.4.66.

(a) $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) = -\frac{1}{2} \times 1 = -\frac{1}{2}$
(b) $g'(2) = f'(4)4 = 1 \times 4 = 4$

3.4.67. $g'(3) = \frac{1}{2\sqrt{f(3)}} f'(3) = \frac{1}{2\sqrt{2}} \left(\frac{-2}{3}\right) = \frac{-1}{3\sqrt{2}}$ (which you can rewrite as $-\frac{\sqrt{2}}{6}$)

3.4.68.

(a) $F'(x) = f'(x^\alpha)\alpha x^{\alpha-1}$
(b) $G'(x) = \alpha \left(f(x)\right)^{\alpha-1} f'(x)$

3.4.72.

(a) $f'(x) = g(x^2) + xg'(x^2)2x = g(x^2) + 2x^2 g'(x^2)$
(b) $f''(x) = g'(x^2)(2x) + 4xg'(x^2) + 2x^2 g''(x^2)2x = 6xg'(x^2) + 4x^3 g''(x^2)$

3.4.80.

(a) $v(t) = s'(t) = -A\omega \sin(\omega t + \delta)$
(b) We want $v(t) = 0$, so $\omega t + \delta = \pi m$, so $t = \frac{\pi m - \delta}{\omega}$ (m is an integer)

3.4.85. $a(t) = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds} v(t)$

3.4.86.

(a) $\frac{dV}{dr}$ is the rate of change of V as the radius r changes, and $\frac{dV}{dt}$ is the rate of change of V as the time t changes
(b) $V(t) = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$

3.4.92.

$$
\begin{aligned}
(f(x)[g(x)]^{-1})' = & f'(x)[g(x)]^{-1} + f(x)(-1)[g(x)]^{-2} g'(x) \\
= & f'(x)g(x)[g(x)]^{-2} - f(x)g'(x)[g(x)]^{-2} \\
= & \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}
$$

Section 3.5: Implicit differentiation

3.5.3. $y' = -\frac{y^2}{x^2}$

3.5.19. $y' = \frac{e^y \sin(x) + \cos(xy)y}{e^y \cos(x) - x\cos(xy)}$

3.5.29. $y = x + \frac{1}{2}$

3.5.30. $(y-1) = \sqrt{3}(x + 3\sqrt{3})$, or $y = \sqrt{3}x + 10$

3.5.32. $y = -2$

3.5.44. First of all, by implicit differentiation:

$$\begin{aligned} \frac{2x}{a^2}+\frac{2yy'}{b^2} &= 0 \\ y'\left(\frac{2y}{b^2}\right) &= -\frac{2x}{a^2} \\ y' &= -\frac{b^2}{a^2}\frac{2x}{2y} \\ y' &= -\frac{b^2}{a^2}\frac{x}{y} \end{aligned}$$

It follows that the tangent line to the ellipse at (x_0, y_0) has slope $-\frac{b^2}{a^2}\frac{x_0}{y_0}$, and since it goes through (x_0, y_0), its equation is:

$$y - y_0 = \left(-\frac{b^2}{a^2}\frac{x_0}{y_0}\right)(x - x_0)$$

And the rest of the problem is just a little algebra!

First of all, by multiplying both sides by $a^2 y_0$, we get:

$$(y - y_0)(a^2 y_0) = -b^2 x_0 (x - x_0)$$

Expanding out, we get:

$$ya^2 y_0 - a^2 (y_0)^2 = -b^2 x_0 x + b^2 (x_0)^2$$

Now rearranging, we have:

$$ya^2 y_0 + b^2 x_0 x = a^2 (y_0)^2 + b^2 (x_0)^2$$

Now dividing both sides by a^2, we get:

$$yy_0 + \frac{b^2}{a^2} x_0 x = (y_0)^2 + \frac{b^2}{a^2}(x_0)^2$$

And dividing both sides by b^2, we get:

$$\frac{yy_0}{b^2} + \frac{x_0 x}{a^2} = \frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2}$$

But now, since (x_0, y_0) is on the ellipse, $\frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2} = 1$, we get:

$$\frac{yy_0}{b^2} + \frac{x_0 x}{a^2} = 1$$

Whence,

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

Which is what we want!

3.5.45. Slope:

$$
\begin{aligned}
\frac{2x}{a^2} - \frac{2yy'}{b^2} =& 0 \\
y'\left(-\frac{2y}{b^2}\right) =& -\frac{2x}{a^2} \\
y' =& \frac{b^2}{a^2}\frac{2x}{2y} \\
y' =& \frac{b^2}{a^2}\frac{x}{y}
\end{aligned}
$$

Equation:
At (x_0, y_0), the slope is $\frac{b^2}{a^2}\frac{x_0}{y_0}$, so the equation of the tangent line at (x_0, y_0) is:

$$y - y_0 = \left(\frac{b^2}{a^2}\frac{x_0}{y_0}\right)(x - x_0)$$

Simplification:
First of all, by multiplying both sides by $a^2 y_0$, we get:

$$(y - y_0)(a^2 y_0) = b^2 x_0 (x - x_0)$$

Expanding out, we get:

$$ya^2 y_0 - a^2 (y_0)^2 = b^2 x_0 x - b^2 (x_0)^2$$

Now rearranging, we have:

$$ya^2 y_0 - b^2 x_0 x = a^2 (y_0)^2 - b^2 (x_0)^2$$

Now dividing both sides by a^2, we get:

$$yy_0 - \frac{b^2}{a^2} x_0 x = (y_0)^2 - \frac{b^2}{a^2}(x_0)^2$$

And dividing both sides by b^2, we get:

$$\frac{yy_0}{b^2} - \frac{x_0 x}{a^2} = \frac{(y_0)^2}{b^2} - \frac{(x_0)^2}{a^2}$$

But now, since (x_0, y_0) is on the hyperbola, $\frac{(x_0)^2}{a^2} - \frac{(y_0)^2}{b^2} = 1$, so $\frac{(y_0)^2}{b^2} - \frac{(x_0)^2}{a^2} = -1$, and we get:

$$\frac{yy_0}{b^2} - \frac{x_0 x}{a^2} = -1$$

Whence,

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$$

3.5.46. Slope:

$$\begin{aligned}\frac{1}{2\sqrt{x}} + y'\left(\frac{1}{2\sqrt{y}}\right) =& 0\\ y'\left(\frac{1}{2\sqrt{y}}\right) =& -\frac{1}{2\sqrt{x}}\\ y' =& -\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}}\\ y' =& -\frac{2\sqrt{y}}{2\sqrt{x}}\\ y' =& -\frac{\sqrt{y}}{\sqrt{x}}\end{aligned}$$

Equation: At (x_0, y_0), the slope is $-\frac{\sqrt{y_0}}{\sqrt{x_0}}$, and so the equation of the tangent line at (x_0, y_0) is:

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

$y-$intercept:
To find the $y-$intercept, set $x = 0$ and solve for y:

$$\begin{aligned}y - y_0 =& -\frac{\sqrt{y_0}}{\sqrt{x_0}}(0 - x_0)\\ y - y_0 =& -\frac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0)\\ y - y_0 =& \sqrt{y_0}\sqrt{x_0}\\ y =& y_0 + \sqrt{y_0}\sqrt{x_0}\end{aligned}$$

$x-$intercept:
To find the $x-$intercept, set $y = 0$ and solve for x:

$$\begin{aligned}0 - y_0 =& -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)\\ -y_0 =& -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)\\ x - x_0 =& -\frac{\sqrt{x_0}}{\sqrt{y_0}}(-y_0)\\ x =& x_0 + \sqrt{x_0}\sqrt{y_0}\end{aligned}$$

Sum:
The sum of the $y-$ and $x-$ intercepts is:

$$(y_0 + \sqrt{y_0}\sqrt{x_0}) + (x_0 + \sqrt{x_0}\sqrt{y_0}) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0$$

But the trick is that:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0})^2 + 2\sqrt{x_0}\sqrt{y_0} + (\sqrt{y_0})^2 = (\sqrt{x_0} + \sqrt{y_0})^2$$

But since (x_0, y_0) is on the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$, we get $\sqrt{x_0} + \sqrt{y_0} = \sqrt{c}$.

And so, finally we get that the sum of the $x-$ and $y-$ intercepts is:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2 = (\sqrt{c})^2 = c$$

3.5.51. $\frac{2}{\sqrt{1-(2x+1)^2}}$ (or $\frac{1}{\sqrt{-(x^2+x)}}$, either answer is fine)

3.5.53. $G'(x) = -\frac{x}{\sqrt{1-x^2}} \arccos(x) - 1$

3.5.54. $\frac{1-\frac{x}{\sqrt{1+x^2}}}{1+(x-\sqrt{1+x^2})^2} = \frac{1}{2(1+x^2)}$

Note: This is a ridiculous simplification, and don't worry about this too much, but here are the steps:

(1) First put everything under a common denominator
(2) Then expand out the $1 + (x - \sqrt{1+x^2})^2$ in the denominator, and simplify and factor out the 2
(3) Then multiply the numerator and the denominator by $\sqrt{1+x^2} + x$ (conjugate form)
(4) Then expand out the $(\sqrt{1+x^2}+x)(1+x^2 - x\sqrt{1+x^2})$-part of the denominator to get $\sqrt{1+x^2}$

3.5.63. Look up the handout 'Proof of the derivative of arccos' on my website!

3.5.77.

(a) $f(f^{-1}(x)) = x$, let $y = f^{-1}(x)$, then $f(y) = x$, so $f'(y)y' = 1$, so $y' = \frac{1}{f(y)} = \frac{1}{f(f^{-1}(x))}$
(b) $\frac{3}{2}$

3.5.80. Let's denote the point of intersection between the ellipse and the tangent line by (a, b).

Then, using implicit differentiation, we can show that the slope of the tangent line is

$$-\frac{a}{4b}$$

Now, let K be the altitude of the lamp, our goal is to find K.

Notice that the same tangent line goes through the points $(-5, 0)$ and $(3, K)$, so by the slope formula, we have:

$$\text{Slope} = \frac{K-0}{3-(-5)} = \frac{K}{8}$$

In particular, since the slope is also equal to $-\frac{a}{4b}$, we have:

So

$$\frac{K}{8} = -\frac{a}{4b}$$

$$K = -8\frac{a}{4b} = -\frac{2a}{b}$$

So all we really need to do to solve this problem is to find $-\frac{2a}{b}$!

Now we also know that the tangent line goes through the points $(-5,0)$ and (a,b), so its slope is $\frac{b-0}{a-(-5)} = \frac{b}{a+5}$, but again we know that its slope is also $-\frac{a}{4b}$, and so we get:

$$\frac{b}{a+5} = -\frac{a}{4b}$$

So cross-multiplying, we have $4b^2 = -(a)(a+5)$, that is $a^2 + 4b^2 = -5a$.

HOWEVER, We also know that (a,b) is on the ellipse, so it satisfies the equation of the ellipse, and so $a^2 + 4b^2 = 5$, whence we get $-5a = 5$, and so $a = -1$.

And plugging $a = -1$ into $a^2 + 4(b)^2 = 5$ and assuming $b > 0$, we get $b = 1$, and so $K = -\frac{2a}{b} = \frac{2}{1} = 2$, and we're done!

Section 3.6: Derivatives of logarithmic functions

3.6.11. $g'(x) = \frac{1}{x\sqrt{x^2-1}}\left(\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}}\right)$

3.6.21. $y' = 2\log_{10}(\sqrt{x}) + 2x\frac{1}{\ln(10)\sqrt{x}} \times \frac{1}{2\sqrt{x}} = 2\log_{10}(\sqrt{x}) + \frac{1}{\ln(10)}$

3.6.41. $y' = \sqrt{\frac{x-1}{x^4+1}}\left(\frac{1}{2(x-1)} + \frac{2x^3}{x^4+1}\right)$

3.6.44. $y' = x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)$

Section 3.7: Rates of change in the natural and social sciences

3.7.4.

(a) $f'(t) = e^{-\frac{t}{2}} - \frac{t}{2}e^{-\frac{t}{2}} = e^{-\frac{t}{2}}\left(1 - \frac{t}{2}\right)$

(b) $f'(3) = e^{-\frac{3}{2}}\left(-\frac{1}{2}\right)$

(c) $t = 2$

(d) When $t < 2$

(e) $f(2) - f(0) + f(2) - f(8) = 2e^{-1} - 0 + 2e^{-1} - 8e^{-4} = 4e^{-1} - 8e^{-4}$

(f) The particle is moving to the right between $t = 0$ and $t = 2$, and then to the left from $t = 2$ to $t = 8$.

(g) $f''(t) = -\frac{1}{2}e^{-\frac{t}{2}}\left(1-\frac{t}{2}\right) + e^{-\frac{t}{2}}\left(-\frac{1}{2}\right) = e^{-\frac{t}{2}}\left(-\frac{1}{2}+\frac{t}{4}-\frac{1}{2}\right) = e^{-\frac{t}{2}}\left(\frac{t}{4}-1\right)$; $f''(3) = e^{-\frac{3}{2}}\left(-\frac{1}{4}\right)$

(h) Use a calculator

(i) Speeding up when $f''(t) > 0$ and $f'(t) > 0$ or when $f''(t) < 0$ and $f'(t) < 0$. But solving those equations reveals that **none** of the two situations can happen! Hence the particle is constantly slowing down!

3.7.8.

(a) First solve for $v(t) = 0$, where $v(t) = \frac{ds}{dt} = 80 - 32t$, you get $\boxed{t = \frac{80}{32} = \frac{5}{2}}$. So the **maximum height** s^* is $s^* = s(\frac{5}{2}) = 200 - 100 = 100$

(b) To find the time t when the ball is 96ft above the ground, we need to solve the equation $s(t) = 96$, and you get $\boxed{t = 2, 3}$, whence $\boxed{v(2) = 80 - 32 \cdot 2 = 16\frac{ft}{s}}$ and $\boxed{v(3) = 80 - 32 \cdot 3 = -16\frac{ft}{s}}$

3.7.17. $f'(x) = 6x =$ linear density at x. $f'(1) = 6$, $f'(2) = 12$, $f'(3) = 18$. The density is highest at 3 and lowest at 1.

3.7.26. First of all, we know two things, namely $f(0) = 20$ and $f'(0) = 12$.

But by the chain rule:

$$f'(t) = -0.7be^{-0.7t}\frac{-a}{(1+be^{-0.7t})^2} = \frac{0.7abe^{-0.7t}}{(1+be^{-0.7t})^2}$$

So from $f(0) = 20$, we get:

$$\frac{a}{1+b} = 20$$

And from $f'(0) = 12$, we get:

$$\frac{0.7ab}{(1+b)^2} = 12$$

From $\frac{a}{1+b} = 20$, we get $a = 20(1+b)$, and plugging this into the second equation, we get:

$$\begin{aligned}\frac{(0.7)(20)(1+b)b}{(1+b)^2} &= 12\\ \frac{14b}{1+b} &= 12\\ 14b &= 12(1+b)\\ 14b &= 12 + 12b\\ 2b &= 12\\ b &= 6\end{aligned}$$

And so $a = 20(1+6) = 20(7) = 140$.

Therefore, we have $\boxed{a = 140}$ and $\boxed{b = 6}$.

Finally, to find out what happens in the long run, we need to calculate $\lim_{t\to\infty} f(t)$. But notice that $\lim_{t\to\infty} e^{-0.7t} = 0$, and so $\lim_{t\to\infty} f(t) = \frac{a}{1+0} = a = \boxed{140}$.

3.7.31.

(a) $C'(x) = 12 - 0.2x + 0.0015x^2$
(b) $C'(200) = 32$; The cost of producing one more yard of a fabric once 200 yards have been produced
(c) $C(201) - C(200) = 32.2005$, which is pretty close to $C'(200)$

Section 3.8: Exponential growth and decay

3.8.6.

(a) **Hint:** Solve the differential equation $y' = ky$ with $y(0) = 361$ and $y(10) = 439$ and find $y(50)$
(b) **Hint:** Solve the differential equation $y' = ky$ with $y(0) = 439$ and $y(20) = 683$ and find $y(39)$ and $y(49)$

3.8.9.

(a) $y(t) = 100e^{\ln\left(\frac{1}{2}\right)\frac{t}{30}} = 100\left(\frac{1}{2}\right)^{\frac{t}{30}}$
(b) $y(100) = 100\left(\frac{1}{2}\right)^{\frac{100}{30}} \approx 9.92$
(c) $t = 30\frac{\ln\left(\frac{1}{100}\right)}{\ln\left(\frac{1}{2}\right)} \approx 199.3$

3.8.11. The problem asks about radioactive decay, so as usual, we have $y' = ky$, so $y(t) = Ce^{kt}$. Now we're given two things: First of all, the half-life is $t = 5730$ years, so $y(5730) = \frac{y(0)}{2} = \frac{C}{2}$. Moreover, we know that at a certain time t^* (we **want to find** t^*), $y(t^*) = 0.74y(0) = 0.74C$. Now even though we don't know what C is, we can still solve for t^*.

The following calculation helps us find k:

$$\begin{aligned}
y(5730) &= \frac{C}{2} \\
Ce^{5730k} &= \frac{C}{2} \\
e^{5730k} &= \frac{1}{2} \\
5730k &= \ln\left(\frac{1}{2}\right) \\
k &= \frac{\ln\left(\frac{1}{2}\right)}{5730}
\end{aligned}$$

Whence $y(t) = Ce^{\frac{\ln\left(\frac{1}{2}\right)}{5730}t} = C\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

Now we're given that $y(t^*) = 0.74C$, and the following calculation helps us solve for t^*:

$$\begin{aligned}
y(t^*) &= 0.74C \\
C(\frac{1}{2})^{\frac{t^*}{5730}} &= 0.74C \\
(\frac{1}{2})^{\frac{t^*}{5730}} &= 0.74 \\
\frac{t^*}{5730}\ln(\frac{1}{2}) &= \ln(0.74) \\
t^* &= 5730\frac{\ln(0.74)}{\ln(\frac{1}{2})} \\
t^* &\approx 2489
\end{aligned}$$

So $\boxed{t^* \approx 2489 \text{ years}}$ (notice how we didn't even need info about C to figure this out!)

3.8.19.

(a) (i) $3000\left(1+\frac{0.05}{1}\right)^{(1)(5)} \approx 3828$

(ii) $3000\left(1+\frac{0.05}{2}\right)^{(2)(5)} \approx 3840$

(iii) $3000\left(1+\frac{0.05}{12}\right)^{(12)(5)} \approx 3850$

(iv) $3000\left(1+\frac{0.05}{52}\right)^{(52)(5)} \approx 3851.61$

(v) $3000\left(1+\frac{0.05}{365}\right)^{(365)(5)} \approx 3852.01$

(vi) $3000e^{0.05(5)} \approx 3852.08$

(b) $A' = 0.05A$, $A(0) = 3000$

Section 3.9: Related rates

3.9.5. $\boxed{\frac{dh}{dt} = \frac{3}{25\pi}}$ (Use $V = \pi r^2 h$)

3.9.13.

1) First of all, let's draw a picture of the situation, and remember to only label things that are **constant!**

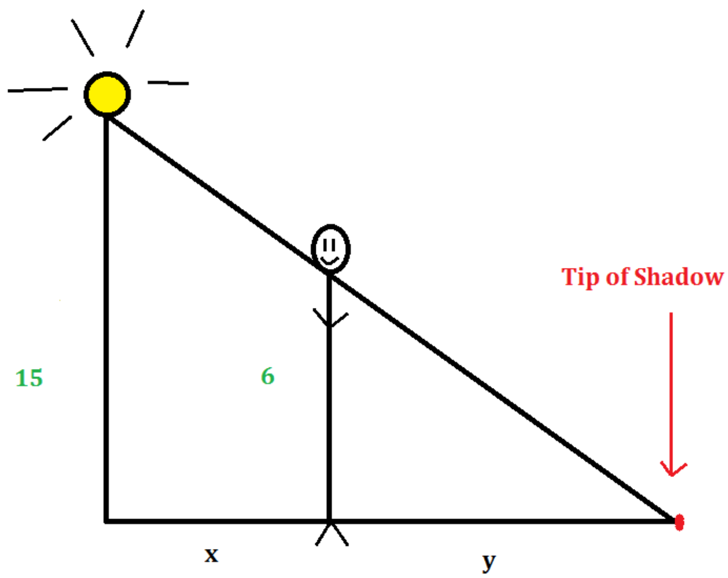
   Here, x is the distance between the street light and the man, and y is the distance between the man and the shadow. Also, let $z = x + y$, the total length of the shadow.
2) We want to figure out z' when $x = 40$.
3) Looking at the picture, we can use the law of similar triangles to conclude:

$$\frac{y}{x+y} = \frac{6}{15}$$

That is:

$$\begin{aligned}
y &= \frac{2}{5}(x+y) \\
\frac{3}{5}y &= \frac{2}{5}x \\
y &= \frac{2}{3}x
\end{aligned}$$

1A/Math 1A Summer/Solution Bank/Street Light.png



It follows that:

$$z = x + y = x + \frac{2}{3}x = \frac{5}{3}x$$

4) Hence $z' = \frac{5}{3}x'$
5) However, we know that $x' = 5$ (because the man is walking with a speed of 5 ft/s).
Hence we get $z' = \frac{5}{3}(5) = \frac{25}{3}$

So $\boxed{z' = \frac{25}{3}}$.

Note: We did not need the fact that $x = 40$!

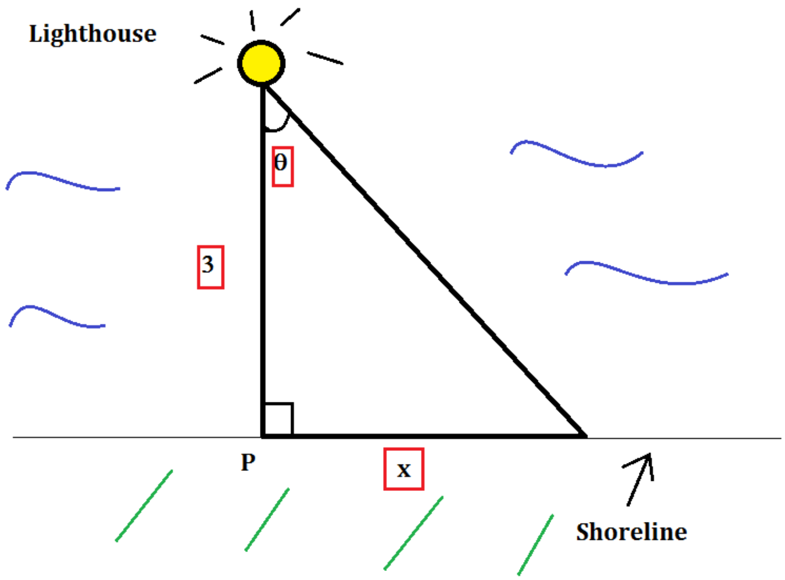
3.9.15. $\boxed{\frac{dD}{dt} = 65mph}$ (use the pythagorean theorem to conclude $D^2 = x^2 + y^2$)

3.9.27. $\boxed{\frac{dh}{dt} = \frac{6}{5\pi}}$ (use the fact that $V = \frac{\pi}{3}r^2 h = \frac{\pi}{12}h^3$ because $h = \frac{r}{2}$)

3.9.40.

1) Again, draw a picture of the situation:
Here, x is the distance between P and the beam of light.

1A/Math 1A Summer/Solution Bank/Lighthouse.png



2) We want to figure out $\frac{dx}{dt}$ when $x = 1$
3) Looking at the picture, because we have info about the derivative of θ (see below), we use the definition of $\tan(\theta)$:

$$\tan(\theta) = \frac{x}{3}$$

So $x = 3\tan(\theta)$

4) Hence $\frac{dx}{dt} = 3\sec^2(\theta)\frac{d\theta}{dt}$
5) First of all, we actually know what $\frac{d\theta}{dt}$ is! Since the lighthouse makes 4 revolutions per minute and one revolution corresponds to 2π, we know that $\frac{d\theta}{dt} = -8\pi$ (think of speed = distance/time, and here time = 1 minute, and 'distance' = 8π, also you put a minus-sign since x is decreasing!).

   Moreover, by drawing the **exact** same picture as above, except with $x = 1$, we can calculate $\sec^2(\theta)$, namely:

$$\sec(\theta) = \frac{hypothenuse}{adjacent} = \frac{\sqrt{10}}{3}$$

And the $\sqrt{10}$ we get from the Pythagorean theorem!

It follows that $\sec^2(\theta) = \left(\frac{\sqrt{10}}{3}\right)^2 = \frac{10}{9}$.
Now we got all of the info we need to conclude the problem:

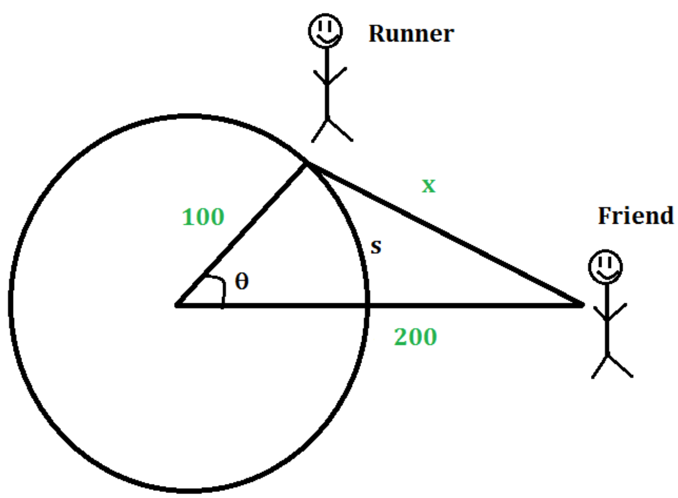
$$\frac{dx}{dt} = 3\sec^2(\theta)\frac{d\theta}{dt}$$
$$\frac{dx}{dt} = 3(\frac{10}{9})(-8\pi)$$
$$\frac{dx}{dt} = -\frac{240\pi}{9}$$
$$\frac{dx}{dt} = -\frac{80\pi}{3}$$

Whence $\boxed{\frac{dx}{dt} = -\frac{80\pi}{3} rad/min}$

3.9.45.

1) As usual, let's draw a picture of the situation:

1A/Math 1A Summer/Solution Bank/Runners.png



Here, x is the distance between the runner and the friend, and s is the length of the arc corresponding to θ

2) We want to figure out $\frac{dx}{dt}$ when $x = 200$
3) Looking at the picture, it looks like we should use the law of cosines (because we have info about θ and about 2 of the 3 sides of the triangle)

$$x^2 = 100^2 + 200^2 - 2(100)(200)\cos(\theta)$$

In other words:

$$x^2 = 50000 - 40000\cos(\theta)$$

4) Hence $2x\frac{dx}{dt} = 40000\sin(\theta)\frac{d\theta}{dt}$
5) First of all $x = 200$ in this case. Also, by definition of a radian, we know that $s = 100\theta$, whence $\frac{ds}{dt} = 100\frac{d\theta}{dt}$. But we're given that $\frac{ds}{dt} = 7m/s$, so $\frac{d\theta}{dt} = \frac{7}{100} = 0.07$.

So all we got to figure out is $\sin(\theta)$

For this, draw the same picture as above, except you let $x = 200$. And in this case, we use the law of cosines **again**:

$$200^2 = 100^2 + 200^2 - 2(100)(200)\cos(\theta)$$

$$-10000 = -40000\cos(\theta)$$

$$\cos(\theta) = \frac{1}{4}$$

Now you can use either the triangle method to figure out what $\sin(\theta)$ is (all you gotta do is calculate $\sin(\cos^{-1}(\frac{1}{4})$, or, even easier, notice that $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$ (this works because $\sin(\theta) > 0$ because we assume that θ is between 0 and $\frac{\pi}{2}$. Hence $\sin(\theta) = \sqrt{1 - \frac{1}{16}} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$.

PHEW!!! Now we have all the info we need to solve the problem:

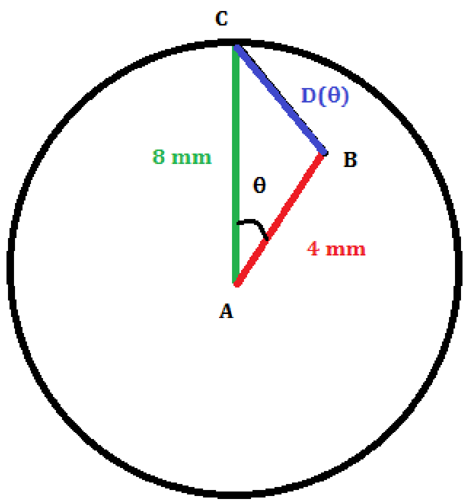
$$\begin{aligned}
2x\frac{dx}{dt} &= 40000\sin(\theta)\frac{d\theta}{dt}\\
2(200)\frac{dx}{dt} &= 40000(\frac{\sqrt{15}}{4})(0.07)\\
400\frac{dx}{dt} &= 700\sqrt{15}\\
\frac{dx}{dt} &= \frac{7}{4}\sqrt{15}
\end{aligned}$$

Hence $\boxed{\frac{dx}{dt} = \frac{7}{4}\sqrt{15}}$

3.9.46. As is usual for related rates problems, let's draw a picture:

Let θ be the angle between the hour hand and the minute hand. Now, what we want to calculate is $D'(\theta)$, where the $'$ indicates differentiation with respect to the time variable.

1A/Math 1A Summer/Solution Bank/Clock.png



How can we relate $D(\theta)$ with what we know? This is easy! We know an angle θ and the lengths of AB and AC in the picture, so let's just use the **law of cosines**. We get:

$$BC^2 = AC^2 + AB^2 - 2 \cdot AC \cdot AB \cdot \cos(\theta)$$

That is:

$$D(\theta)^2 = 8^2 + 4^2 - 2 \cdot 8 \cdot 4 \cdot \cos(\theta)$$

Which you can write as:

$$D(\theta)^2 = 80 - 64\cos(\theta)$$

Now differentiate with respect to time!
We get:

$$2D(\theta)D'(\theta) = 64\sin(\theta)\frac{d\theta}{dt}$$

And now, all we need to do is to plug in everything we know!

First of all $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ (basically, the whole circle corresponds to 2π, and so $\frac{1}{12}$ of the circle corresponds to $\frac{2\pi}{12}$).

In particular, $\boxed{\sin(\theta) = \sin(\frac{\pi}{6}) = \frac{1}{2}}$.

Now, for $d(\theta)$, we use the law of cosines again, this time with the value $\theta = \frac{\pi}{6}$:

$$D(\frac{\pi}{6})^2 = 80 - 64 \cdot \cos(\frac{pi}{6}) = 80 - 64\frac{\sqrt{3}}{2} = 80 - 32\sqrt{3}$$

So, taking square roots, we get $\boxed{d(\frac{\pi}{6}) = \sqrt{80 - 32\sqrt{3}} = 4\sqrt{5 - 2\sqrt{3}} \text{ mm}}$.

Finally, we need to compute $\frac{D\theta}{dt}$. But think about it! In 12 hours, $\theta = 2\pi$, so the speed of θ should be $\frac{2\pi}{12} = \frac{\pi}{6} = -\frac{11\pi}{6}$ rad/h (notice that θ is decreasing, so we wanted a negative answer!)

Finally, we have all our information to get our final answer:

$$\begin{aligned}
2D(\theta)D'(\theta) &= 64\sin(\theta)\frac{D\theta}{dt} \\
2 \cdot (4\sqrt{5 - 2\sqrt{3}}) \cdot D'\left(\frac{\pi}{6}\right) &= 64 \cdot \frac{1}{2} \cdot \frac{-11\pi}{6} \\
2 \cdot (4\sqrt{5 - 2\sqrt{3}}) \cdot D'\left(\frac{\pi}{6}\right) &= 32 \cdot \frac{-11\pi}{6} \\
D'\left(\frac{\pi}{6}\right) &= 16 \cdot \frac{\frac{-11\pi}{6}}{4\sqrt{5 - 2\sqrt{3}}} \\
D'\left(\frac{\pi}{6}\right) &= 4 \cdot \frac{\frac{-11\pi}{6}}{\sqrt{5 - 2\sqrt{3}}} \\
D'\left(\frac{\pi}{6}\right) &= -\frac{22\pi}{3\sqrt{5 - 2\sqrt{3}}} \\
D'\left(\frac{\pi}{6}\right) &\approx -18.55mm/h
\end{aligned}$$

So our final answer is $\boxed{D'\left(\frac{\pi}{6}\right) = -\frac{22\pi}{3\sqrt{5-2\sqrt{3}}} \text{ mm/h} \approx -18.55mm/h}$

Section 3.10: Linear approximations and differentials

3.10.2. $L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$

3.10.11.

(a) $dy = (2x\sin(2x) + 2x^2\cos(2x))dx$

(b) $dy = \frac{1}{\sqrt{1+t^2}}\left(\frac{t}{\sqrt{1+t^2}}\right)dt = \frac{t}{1+t^2}dt$

3.10.15.

(a) $dy = \frac{1}{10}e^{\frac{x}{10}}dx$

(b) $dy = \frac{1}{10}(0.1) = 0.01$

3.10.21. $\Delta(y) = y(5) - y(4) = \frac{2}{5} - \frac{2}{4} = -\frac{1}{10} = -0.1$
$dy = -\frac{2}{4^2}(1) = -\frac{1}{8} = -0.125$

3.10.35. $l = 2\pi r = 84$, so $r = \frac{84}{2\pi} = \frac{42}{\pi}$. We know $dl = 0.5$, so $2\pi dr = 0.5$, so $dr = \frac{0.5}{2\pi} = \frac{1}{4\pi}$

(a) $S = 4\pi r^2$, so $dS = 8\pi r dr = 8\pi \frac{42}{\pi}\frac{1}{4\pi} = \frac{84}{\pi}$. Also the relative error is $\frac{dS}{S} = \frac{8\pi r dr}{4\pi r^2} = \frac{2dr}{r} = \frac{1}{2\pi} \times \frac{\pi}{42} = \frac{1}{84} \approx 0.012$

(b) $V = \frac{4}{3}\pi r^3$, so $dV = 4\pi r^2 dr = 4\pi \frac{42^2}{\pi^2} \times \frac{1}{4\pi} = \frac{1764}{\pi^2} \approx 179$. Also the relative error is $\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = \frac{3dr}{r} = \frac{\frac{3}{4\pi}}{\frac{42}{\pi}} = \frac{3}{168} = \frac{1}{56} \approx 0.018$

3.10.40. $dF = 4kR^3 dR$, so:

$$\frac{dF}{F} = \frac{4kR^3 dR}{F} = \frac{4kR^3 dR}{F} = \frac{4kR^3 dR}{kR^4} = 4\frac{dR}{R}$$

And when $\frac{dR}{R} = 0.05$, $\frac{dF}{F} = 4(0.05) = 0.2$

3.10.43.

(a) $L(x) = f(1) + f'(1)(x-1) = 5 - 1(x-1) = 6 - x$
$f(0.9) \approx L(0.9) = 5 - (0.9 - 1) = 5.1$
$f(1.1) \approx L(1.1) = 5 - (1.1 - 1) = 4.9$

(b) Notice that $(f')'(x) < 0$, hence $f''(x) < 0$, so f is concave down (Think for example $\sqrt{x}$, we'll discuss that more in section 4.3), which means that the linear approximations will be overestimates (in other words, the tangent line to f at 1 will be above the graph of f; again, think of the case $\sqrt{x}$)

Section 3.11: Hyperbolic functions

3.11.9.

$$\cosh(x) + \sinh(x) = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x} + e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x$$

3.11.11. Just expand out the right-hand-side and use the fact that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(y) = \frac{e^y - e^{-y}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and $\sinh(y) = \frac{e^y - e^{-y}}{2}$

3.11.15.

$$2\sinh(x)\cosh(x) = 2\frac{e^x - e^{-x}}{2}\frac{e^x + e^{-x}}{2} = \frac{2}{4}(e^x - e^{-x})(e^x + e^{-x}) = \frac{1}{2}(e^{2x} - e^{-2x}) = \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x)$$

3.11.21. $\sinh(x) = \frac{4}{3}$ (use the fact that $\cosh^2(x) - \sinh^2(x) = 1$ and the fact that $\sinh(x) > 0$ when $x > 0$).

Then you get $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{4}{3}}{\frac{5}{3}} = \frac{4}{5}$, $sech(x) = \frac{1}{\cosh(x)} = \frac{3}{5}$, etc.

3.11.23(a). 1 (use $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and factor out e^x from the numerator and the denominator)

3.11.26. This is very similar to example 3 on page 257. However, there is a subtle point involved, check out the document 'Subtle point in 3.11.26' for more info!

3.11.29(a)(b). This is also very similar to example 4 on page 257. For (a), use the fact that $\cosh(\cosh^{-1}(x))^2 - \sinh(\cosh^{-1}(x))^2 = 1$ and $\cosh(\cosh^{-1}(x)) = x$. Also, you'll need to fact that $\sinh(\cosh^{-1}(x)) \geq 0$ (and this is because $\cosh^{-1}(x) \geq 0$ by definition, and $\sinh(x) \geq 0$ if $x \geq 0$). (b) is even easier, use the fact that: $1 - \tanh(\tanh^{-1}(x))^2 = sech(\tanh^{-1}(x))^2$ and $\tanh(\tanh^{-1}(x)) = x$.

3.11.31. $f'(x) = \sinh(x) + x\cosh(x) - \sinh(x) = x\cosh(x)$

3.11.47. $y' = \frac{1}{1+\tanh^2(x)}(sech^2(x)) = \frac{sech^2(x)}{1+\tanh^2(x)}$

Chapter 3 – Review

TRUE–FALSE.

(1) **TRUE**
(2) **FALSE**
(3) **TRUE**
(4) **TRUE**
(5) **FALSE** ($\frac{f'(\sqrt{x})}{2\sqrt{x}}$)
(6) **FALSE** (e^2 is a constant, so 0)
(7) **FALSE** ($y' = \ln(10)10^x$, exponential rule)
(8) **FALSE** ($\ln 10$ is a constant, so 0)
(9) **FALSE** ($2\tan(x)\sec^2(x)$)
(10) **FALSE** ($(2x+1)\frac{x^2+x}{|x^2+x|}$; Write $\left|x^2+x\right| = \sqrt{(x^2+x)^2}$ and use the chain rule)
(11) **TRUE**
(12) **TRUE** (f is a polynomial of degree 30, so its 31st derivative is 0)
(13) **TRUE** (By the quotient rule and (11))
(14) **FALSE** ($y - 4 = -4(x+2)$; it's not even the equation of a line!)
(15) **TRUE** ($= g'(2) = 5(2)^4 = 80$)

3.R.68.

(a) $\sin(2x) = 2\sin(x)\cos(x)$
(b) $\cos(x+a) = \cos(x)\cos(a) - \sin(x)\sin(a)$ (The important thing here is that you differentiate with respect to x, leaving a constant)

3.R.87.

$$v(t) = s'(t) = -Ace^{-ct}\cos(\omega t+\delta) - A\omega e^{-ct}\sin(\omega t+\delta) = -Ae^{-ct}\left(c\cos(\omega t+\delta) + \omega\sin(\omega t+\delta)\right)$$

$$a(t) = v'(t) = Ace^{-ct}\left(c\cos(\omega t+\delta) + \omega\sin(\omega t+\delta)\right) - Ae^{-ct}\left(-c\omega\sin(\omega t+\delta) + \omega^2\cos(\omega t+\delta)\right)$$

3.R.94. $y(t) = 100 \times 2^{-\frac{t}{5.24}}$

(a) $y(20) = 100 \times 2^{\frac{-20}{5.24}} \approx 7.1$ mg
(b) $t = \frac{5.24\ln(100)}{\ln(2)} \approx 34.81$ years

3.R.99.

(1) $D^2 = x^2 + y^2$ (Typical Pythagorean theorem problem; Draw a right triangle in the shape of an L, and let x be the bottom side, y be the left-hand-side, and D be the hypothenus)
(2) $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$
(3) $x = 3 \times 15 = 45$ (velocity $\times$ time), $y = 45 + 5 \times 3 = 60$ (initial height + velocity $\times$ time), $\frac{dx}{dt} = 15$, $\frac{dy}{dt} = 5$, and $D = \sqrt{x^2+y^2} = \sqrt{45^2+60^2} = \sqrt{5625} = 75$ which gives:

$$\frac{dD}{dt} = \frac{45 \times 15 + 60 \times 5}{75} = 13$$

3.R.101.

(1) $\tan(\theta) = \frac{400}{x}$ (Typical trigonometry-problem. Draw another triangle in the shape of an L, let 400 be the left-hand-side, x be the bottom, and the angle on the right be θ)
(2) $\sec^2(\theta)\frac{d\theta}{dt} = -\frac{400}{x^2}\frac{dx}{dt}$
(3) $x = 400\sqrt{3}$ (redraw the same triangle, but this time with $\theta = \frac{\pi}{6}$), $\frac{d\theta}{dt} = -0.25$, and $\theta = \frac{\pi}{6}$, which gives:

$$\frac{dx}{dt} = 400$$

3.R.105. The area of the window is given by $y = x^2 + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2$. Then:

$$dy = \left(1 + \frac{\pi}{8}\right)2xdx = \left(1 + \frac{\pi}{8}\right)(120)(0.1) = 12 + \frac{3\pi}{2} \approx 16.71$$

3.R.111. $f'(2x) = \frac{x^2}{2}$, so $f'(x) = \frac{\left(\frac{x}{2}\right)^2}{2} = \frac{x^2}{8}$

Section 4.1: Maximum and Minimum Values

4.1.6.

- Absolute maximum: Does not exist (**NOT** 5)
- Absolute minimum: $f(4) = 1$
- Local minimum: $f(2) = 2$, $f(4) = 1$
- Local maximum: $f(3) = 4$, $f(6) = 3$

4.1.38. 0 (g' does not exist), $-\frac{1}{2}$ (makes $g'(c) = 0$)

4.1.39. 0 (F' does not exist), 4, $\frac{8}{7}$ (makes $F'(c) = 0$)

4.1.41. $f'(\theta) = -2\sin(\theta) + 2\sin(\theta)\cos(\theta)$, which gives $\theta = \pi m$ or $\theta = 2\pi m$, which can be just written as $\boxed{\theta = \pi m}$ (m is an integer)

4.1.43. $f'(x) = 2xe^{-3x} - 3x^2e^{-3x}$, which gives $\boxed{x = 0}$ and $\boxed{x = \frac{2}{3}}$

4.1.51. $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$, which gives $x = -1, 0, 2$. Candidates: $f(-2) = 33$, $f(3) = 28$ (endpoints), $f(0) = 1$, $f(-1) = -4$, $f(2) = -31$. Absolute maximum: $\boxed{f(-2) = 33}$, Absolute minimum: $\boxed{f(2) = -31}$

4.1.57.

1) Evaluate f at the endpoints 0 and $\frac{\pi}{2}$

$$f(0) = 2 + 0 = 2,\ f\left(\tfrac{\pi}{2}\right) = 0 + 0 = 0$$

2) Find the critical numbers of f

$$f'(t) = -2\sin(t) + 2\cos(2t)$$

$$\begin{aligned} f'(t) &= 0 \\ -2\sin(t) + 2\cos(2t) &= 0 \\ \sin(t) &= \cos(2t) \end{aligned}$$

Here comes the tricky part! This seems impossible to solve, but ideally we'd like to write the right-hand-side just in terms of $\sin(x)$ in order to have a shot at solving this!

Start with $\cos(2t) = \cos^2(t) - \sin^2(t)$ (the double-angle formula for cos).
Moreover $\cos^2(t) = 1 - \sin^2(t)$ (because $\cos^2(t) + \sin^2(t) = 1$)
So we get $\cos(2t) = 1 - \sin^2(t) - \sin^2(t) = 1 - 2\sin^2(t)$. So our original equation becomes:

$$\sin(t) = 1 - 2\sin^2(t)$$

which you can rewrite as $2\sin^2(t) + \sin(t) - 1 = 0$.

This again looks impossible to solve, but notice that this is just a quadratic equation in $\sin(t)$! So let $X = \sin(t)$, then we get:

$$2X^2 + X - 1 = 0$$

And using the quadratic formula (or your factoring skills), we get:

$$(2X - 1)(X + 1) = 0$$

So $X = \frac{1}{2}$ or $X = -1$. That is, $\sin(t) = \frac{1}{2}$ or $\sin(t) = -1$.
HOWEVER, remember that we're only focusing on $[0, \frac{\pi}{2}]$, so in particular $\sin(t) = \frac{1}{2}$ has only one solution in $[0, \frac{\pi}{2}]$, namely $t = \frac{\pi}{6}$, and $\sin(t) = -1$ has **NO** solution in $[0, \frac{\pi}{2}]$.

It follows that the only critical number of f in $[0, \frac{\pi}{2}]$ is $\boxed{t = \frac{\pi}{6}}$ (there are no numbers where f is not differentiable, so in fact those are all the critical numbers).

And we get $f(\frac{\pi}{6}) = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

3) Compare all the candidates you have:

Our candidates are $f(0) = 2$, $f(\frac{\pi}{2}) = 0$ and $f(\frac{\pi}{6}) = \frac{3\sqrt{3}}{2} \approx 2.59$.

Hence, the absolute minimum of f on $[0, \frac{\pi}{2}]$ is $\boxed{f(\frac{\pi}{2}) = 0}$ (the smallest candidate), and the absolute maximum of f on $[0, \frac{\pi}{2}]$ is $\boxed{f(\frac{\pi}{6}) = \frac{3\sqrt{3}}{2}}$ (the largest candidate)

Section 4.2: The Mean Value Theorem

4.2.6. $f(0) = f(\pi) = 0$, but $f'(x) = \sec^2(x) > 0$. This does not contradict Rolle's Theorem because f is not continuous on $[0, \pi]$ (it is discontinuous at $\frac{\pi}{2}$.

4.2.11. $\ln(x)$ is continuous on $[1, 4]$, differentiable on $(1, 4)$; $c = \frac{3}{\ln(4)}$

4.2.17. Let $f(x) = 2x - 1 - \sin(x)$

At least one root: $f(0) = -1 < 0$ and $f(\pi) = 2\pi - 1 > 0$ and f is continuous, so by the **Intermediate Value Theorem (IVT)** the equation has at least one root.

At most one root: Suppose there are two roots a and b. Then $f(a) = f(b) = 0$, so by **Rolle's Theorem** there is at least one $c \in (a, b)$ such that $f'(c) = 0$. But $f'(c) = 2 - \cos(c) \neq 0$, which is a contradiction, and hence the equation has at most one root.

4.2.18. Let $f(x) = x^3 + e^x$

At least one root: $f(-1) = -1 + e^{-1} < 0$ and $f(0) = 0 + e^0 = 1 > 0$ and f is continuous, so by the **Intermediate Value Theorem (IVT)** the equation has at least one root.

At most one root: Suppose there are two roots a and b. Then $f(a) = f(b) = 0$, so by **Rolle's Theorem** there is at least one $c \in (a, b)$ such that $f'(c) = 0$. But $f'(c) = 3c^2 + e^c \geq e^c > 0$, so $f'(c) \neq 0$, which is a contradiction, and hence the equation has at most one root.

4.2.23. By the MVT, $\frac{f(4)-f(1)}{4-1} = f'(c)$ for some c in $(1, 4)$. Solving for $f(4)$ and using $f(1) = 10$, we get $f(4) = 3f'(c) + 10 \geq 6 + 10 = 16$.

4.2.29. This is equivalent to showing:

$$\left| \frac{\sin(a) - \sin(b)}{a - b} \right| \leq 1$$

Which is the same as:

$$\left| \frac{\sin(b) - \sin(a)}{b - a} \right| \leq 1$$

Which is the same as:

$$-1 \leq \frac{\sin(b) - \sin(a)}{b - a} \leq 1$$

But by the MVT applied to $f(x) = \sin(x)$, we get:

$$\frac{\sin(b) - \sin(a)}{b - a} = \cos(c)$$

for some c in (a, b). However, $-1 \leq \cos(c) \leq 1$, and so we're done!

4.2.32. Let $f(x) = 2\sin^{-1}(x)$, $g(x) = \cos^{-1}(1-2x^2)$.

Then $f'(x) = \frac{2}{\sqrt{1-x^2}}$ and:

$$g'(x) = -\frac{-4x}{\sqrt{1-(1-2x^2)^2}} = \frac{4x}{\sqrt{1-1+4x^2-4x^4}} = \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}} = f'(x)$$

(you get this by factoring out $4x^2$ out of the square root. This works because $x \geq 0$)

Hence $f'(x) = g'(x)$, so $f(x) = g(x) + C$

To get C, plug in $x = 0$, so $f(0) = g(0) + C$. But $f(0) = g(0) = 0$, so $C = 0$, whence $\boxed{f(x) = g(x)}$

4.2.34. Let $f(t)$ be the speed at time t. By the MVT with $a = 2:00$ and $b = 2:10$, we get:

$$\frac{f(2:10) - f(2:00)}{2:10 - 2:00} = f'(c)$$

But $2:10 - 2:00 = 10$ minutes $= \frac{1}{6}$ h, so:

$$\frac{50-30}{\frac{1}{6}} = f'(c)$$

Whence: $\boxed{f'(c) = 120}$ for some c between $2:00$ pm and $2:10$ pm. But $f'(c)$ is the acceleration at time c, and so we're done!

4.2.36. This is again a proof by contradiction!

Suppose f has (at least) two fixed points a and b.

Then, by definition of a fixed point, $f(a) = a$, and $f(b) = b$.

However, by the **Mean Value Theorem**, there is a c in (a, b) such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Now using the fact that $f(b) = b$ and $f(a) = a$, we get:

$$\frac{b-a}{b-a} = f'(c)$$

So

$$1 = f'(c)$$

That is, $f'(c) = 1$. However, by assumption, $f'(x) \neq 1$ for all x, so in particular setting $x = c$ gives $f'(c) \neq 1$.

but since $f'(c) = 1$, we get $1 \neq 1$, which is a contradiction!

Hence f has at most one fixed point!

Section 4.3: How derivatives affect the shape of a graph

4.3.2.

(a) $(0,1)\cup(3,7)$
(b) $(1,3)$
(c) $(2,4)\cup(5,7)$
(d) $(0,2)\cup(4,5)$
(e) $(2,2)$, $(4,2.5)$, $(5,4)$

4.3.9.

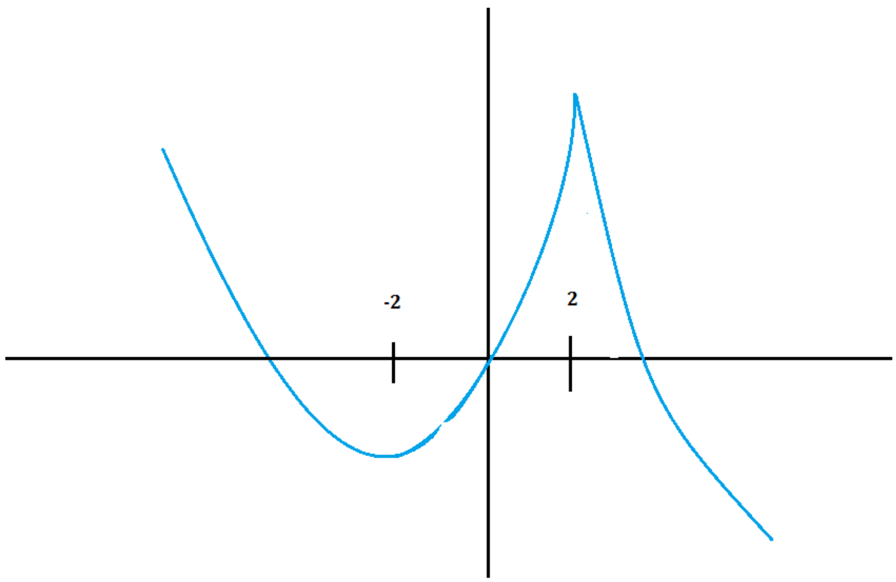
(a) $f'(x)=6x^2+6x-36=6(x-2)(x+3)$; $\nearrow$ on $(-\infty,-3)\cup(2,\infty)$, $\searrow$ on $(-3,2)$
(b) Local max: $f(-3)=81$; Local min: $f(2)=-44$
(c) $f''(x)=12x+6$; CU on $(-\frac{1}{2},\infty)$, CD on $(-\infty,\frac{-1}{2})$, IP $(-\frac{1}{2},f(-0.5)=\frac{37}{2})$

4.3.13.

(a) $f'(x)=\cos(x)-\sin(x)$; $\nearrow$ on $(0,\frac{\pi}{4})\cup(\frac{5\pi}{4},\infty)$, $\searrow$ on $(\frac{\pi}{4},\frac{5\pi}{4})$
(b) Local max: $f(\frac{\pi}{4})=\sqrt{2}$; Local min: $f(\frac{5\pi}{4})=-\sqrt{2}$
(c) $f''(x)=-\sin(x)+\cos(x)$; CU on $(\frac{3\pi}{4},\frac{7\pi}{4})$, CD on $(0,\frac{3\pi}{4})\cup(\frac{7\pi}{4},2\pi)$, IP $(\frac{3\pi}{4},0),(\frac{7\pi}{4},0)$

4.3.27. A possible graph looks like this:

1A/Math 1A Summer/Solution Bank/Concave-Kink.png



4.3.33.

(a) $f'(x) = 3x^2 - 12$, $\nearrow$ on $(\infty, -2) \cup (2, \infty)$, $\searrow$ on $(-2, 2)$
(b) Local min: $f(2) = -14$, Local max: $f(-2) = 18$
(c) $f''(x) = 6x$; CD on $(-\infty, 0)$, CU on $(0, \infty)$; Inflection point at $(0, 2)$
(d) Draw the graph!

4.3.43.

(a) $f'(\theta) = -2\sin(\theta) - 2\cos(\theta)\sin(\theta) = -2\sin(\theta)(1 + \cos(\theta))$; $\nearrow$ on $(\pi, 2\pi)$, $\searrow$ on $(0, \pi)$
(b) Local min: $f(\pi) = -1$, no local max.
(c) $f''(x) = -2\cos(\theta) + 2\sin^2(\theta) - 2\cos^2(\theta) = -2\cos(\theta) + 2 - 4\cos^2(\theta) = -4(\cos^2(\theta) - \frac{\cos(\theta)}{2} - \frac{1}{2}) = -4(\cos(\theta) + 1)(\cos(\theta) - \frac{1}{2})$; CU on $(\frac{\pi}{3}, \frac{5\pi}{3})$, CD on $(0, \frac{\pi}{3}) \cup (\frac{5\pi}{3}, 2\pi)$, IP $(\frac{\pi}{3}, \frac{5}{4}), (\frac{5\pi}{3}, \frac{5}{4})$
(d) Draw the graph!

4.3.45.

(a) VA: $x = 0$, HA: $y = 1$ (at $\pm\infty$)
(b) $f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = \frac{2-x}{x^3}$; $\searrow$ on $(\infty, 0) \cup (2, \infty)$, $\nearrow$ on $(0, 2)$
(c) Local maximum at $(2, \frac{5}{4})$, No local minimum

(d) $f''(x) = \frac{-6x^2+2x^3}{x^6} = \frac{-6+2x}{x^4} = \frac{2x-6}{x^4}$; CD on $(-\infty, 0) \cup (0, 3)$, CU on $(3, \infty)$; IP at $(3, \frac{11}{9})$
(e) Draw the graph!

4.3.49.

(a) No VA; HA: $y = 0$ (at $\pm\infty$)
(b) $f'(x) = (-2x)e^{-x^2}$, $\nearrow$ on $(-\infty, 0)$, $\searrow$ on $(0, \infty)$
(c) Local maximum at $(0, 1)$, no local minimum
(d) $f''(x) = (-2+4x^2)e^{-x^2} = 2(2x^2-1)e^{-x^2}$; CU on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$; IP at $\left(\pm\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$
(e) Draw the graph!

4.3.77. Let $f(x) = \tan(x) - x$, then $f'(x) = \sec^2(x) - 1 = 1 + \tan^2(x) + 1 - 1 = \tan^2(x) > 0$ on $\left(0, \frac{\pi}{2}\right)$, hence f is increasing on $\left(0, \frac{\pi}{2}\right)$. In particular, $f(x) > f(0) = 0$, so $\tan(x) - x > 0$, so $\tan(x) > x$ on $\left(0, \frac{\pi}{2}\right)$

Section 4.4: L'Hopital's Rule

4.4.3.

(a) No, $-\infty$
(b) Yes, $\infty - \infty$
(c) No, ∞

4.4.4.

(a) Yes, 0^0
(b) No, 0
(c) Yes, 1^∞
(d) Yes, ∞^0
(e) No, ∞
(f) Yes, ∞^0

4.4.11.

$$\lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{\cos(x)}{1-\sin(x)} = \lim_{x\to\left(\frac{\pi}{2}\right)^+} \frac{-\sin(x)}{-\cos(x)} = \frac{-1}{-0^-} = \frac{-1}{0^+} = -\infty$$

4.4.16.

$$\begin{aligned}\lim_{\theta\to\frac{\pi}{2}} \frac{1-\sin(\theta)}{\csc(\theta)} &= \lim_{\theta\to\frac{\pi}{2}} \frac{-\cos(\theta)}{\frac{-\cos(\theta)}{\sin^2(\theta)}} \text{(l'Hopital's rule)} \\ &= \lim_{\theta\to\frac{\pi}{2}} \sin^2(\theta) \text{(cancelling out)} \\ &= 1\end{aligned}$$

4.4.17.

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

4.4.27.

$$\lim_{x\to 0} \frac{\tanh(x)}{\tan(x)} = \lim_{x\to 0} \frac{sech^2(x)}{\sec^2(x)} = 1$$

4.4.29.

$$\lim_{x\to 0}\frac{\sin^{-1}(x)}{x}=\lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}}{1}=1$$

4.4.45.

$$\lim_{x\to\infty}x^3e^{-x^2}=\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=\lim_{x\to\infty}\frac{3x^2}{2xe^{x^2}}=\frac{3}{2}\lim_{x\to\infty}\frac{x}{e^{x^2}}=\frac{3}{2}\lim_{x\to\infty}\frac{1}{2xe^{x^2}}=\frac{3}{2}\times 0=0$$

4.4.49.

$$\lim_{x\to 1}\frac{x}{x-1}-\frac{1}{\ln(x)}=\lim_{x\to 1}\frac{x\ln(x)-(x-1)}{(x-1)\ln(x)}=\lim_{x\to 1}\frac{\ln(x)+1-1}{\ln(x)+1-\frac{1}{x}}=\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}=\frac{1}{2}$$

4.4.53.

$$\lim_{x\to\infty}x-\ln(x)=\lim_{x\to\infty}x(1-\frac{\ln(x)}{x})=\infty\times(1-0)=\infty$$

4.4.58.

1) Let $y=\left(1+\frac{a}{x}\right)^{bx}$
2) $\ln(y)=bx\ln(1+\frac{a}{x})$
3)

$$\lim_{x\to\infty}\ln(y)=\lim_{x\to\infty}bx\ln(1+\frac{a}{x})=\lim_{x\to\infty}\frac{\ln(1+\frac{a}{x})}{\frac{1}{bx}}=\lim_{x\to\infty}\frac{(\frac{1}{1+\frac{a}{x}})(-\frac{a}{x^2})}{(-\frac{1}{x^2})(\frac{1}{b})}=\lim_{x\to\infty}\frac{ab}{1+\frac{a}{x}}=ab$$

4) So $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=e^{ab}$

4.4.61.

1) Let $y=x^{\frac{1}{x}}$
2) Then $\ln(y)=\frac{\ln(x)}{x}$
3) So $\lim_{x\to\infty}\ln(y)=\lim_{x\to\infty}\frac{\ln(x)}{x}=0$
4) Hence $\lim_{x\to\infty}x^{\frac{1}{x}}=\lim_{x\to\infty}y=e^0=1$

4.4.76.

(a) Here, you want to calculate $\lim_{t\to\infty}$, so **treat t as our x, and leave everything else as a constant!!!!**. In particular, we get:

$$\lim_{t\to\infty}e^{-\frac{ct}{m}}=0\qquad\text{because } c>0 \text{ and } m>0$$

So, we get:

$$\lim_{t\to\infty}v=\frac{mg}{c}(1-0)=\frac{mg}{c}$$

(b) Here, we let $c\to 0^+$, so **we treat c as our x, and leave everything else as a constant!**.

Then, we have:

$$\begin{aligned}
\lim_{c\to 0^+} v = \lim_{c\to 0^+} mg\left(\frac{1-e^{-\frac{ct}{m}}}{c}\right) \\
&= \lim_{c\to 0^+} mg\left(\frac{-\left(-\frac{t}{m}\right)e^{-\frac{ct}{m}}}{1}\right) \qquad \text{(by l'Hopital's rule)} \\
&= mg\left(\frac{\left(\frac{t}{m}\right)e^{0}}{1}\right) \\
&= mg\left(\frac{\left(\frac{t}{m}\right)1}{1}\right) \\
&= \frac{mgt}{m} \\
&= gt
\end{aligned}$$

4.4.77. All you gotta do is show that:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

1) Let $y = \left(1+\frac{r}{n}\right)^{nt}$
2) $\ln(y) = nt\ln(1+\frac{r}{n})$
3) The important thing to realize is that you're taking the limit as n goes to ∞, which means that r and t are **constants**!

$$\lim_{n\to\infty}\ln(y) = \lim_{n\to\infty} nt\ln(1+\frac{r}{n}) = \lim_{n\to\infty}\frac{\ln(1+\frac{r}{n})}{\frac{1}{nt}} = \lim_{n\to\infty}\frac{\frac{\frac{-r}{n^2}}{1+\frac{r}{n}}}{-\frac{1}{n^2t}} = \lim_{n\to\infty}\frac{\frac{rn^2t}{n^2}}{1+\frac{r}{n}} = \lim_{n\to\infty}\frac{rt}{1+\frac{r}{n}} = \frac{rt}{1+0} = rt$$

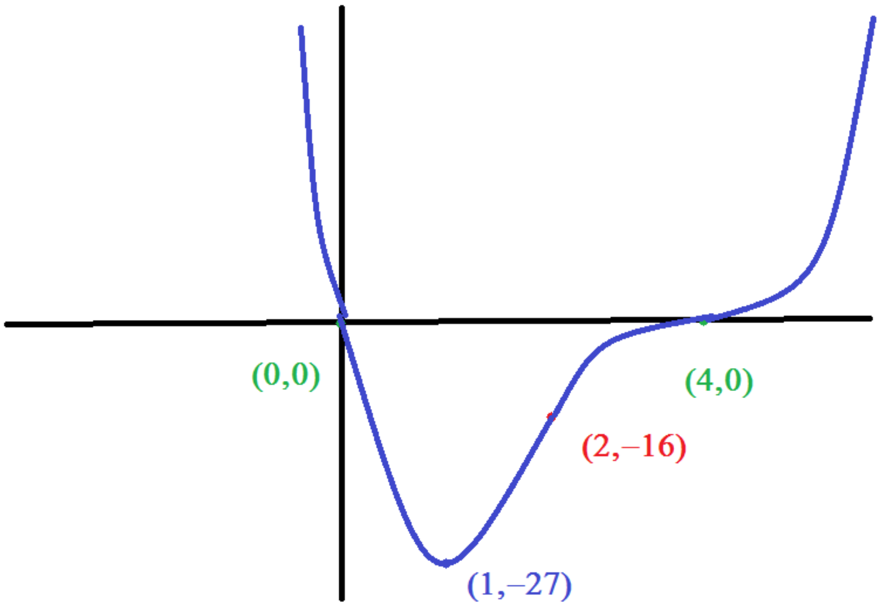
4) So $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$, and hence $\lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0e^{rt}$

Section 4.5: Summary of curve sketching

4.5.5.

D : $\mathbb{R}$
I : $x-$intercepts: $0, 4$, $y-$intercept: 0
S : None
A : None, but $\lim_{x\to\pm\infty} f(x) = \infty$
I : $f'(x) = (x-4)^3 + 3x(x-4)^2 = (x-4)^2(x-4+3x) = (x-4)^2(4x-4) = 4(x-4)^2(x-1)$; f is decreasing on $(-\infty, 1)$ and increasing on $(1,\infty)$; Local minimum $f(1) = -27$
C : $f''(x) = 8(x-4)(x-1) + 4(x-4)^2 = 4(x-4)(2x-2+x-4) = 4(x-4)(3x-6) = 12(x-4)(x-2)$; f is concave up on $(-\infty, 2)$, concave down on $(2,4)$, and concave up on $(4,\infty)$. Inflection points: $(2,-16)$, $(4,0)$

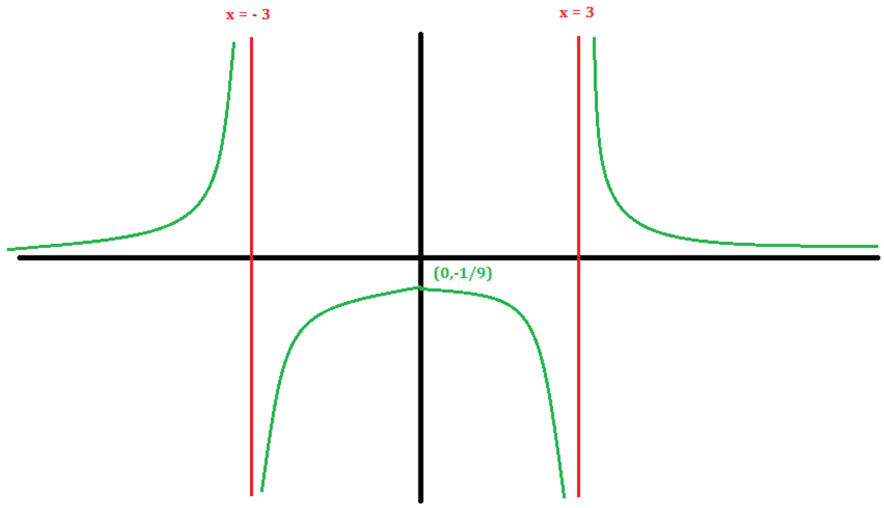
1A/Math 1A - Fall 2013/Homeworks/Quartic.png



4.5.13.

D : $\mathbb{R} - \{\pm 3\}$
I : No $x-$intercepts, $y-$intercept: $y = -\frac{1}{9}$
S : f is even
A : Horizontal Asymptote $y = 0$ (at $\pm\infty$), Vertical Asymptotes $x = \pm 3$
I : $f'(x) = -\frac{2x}{(x^2-9)^2}$; f is increasing on $(-\infty, -3) \cup (-3, 0)$ and decreasing on $(0, 3) \cup (3\infty)$. Local maximum of $\frac{-1}{9}$ at 0.
C : $f''(x) = 6\frac{x^2+3}{(x^2-9)^3}$; f is concave up on $(-\infty, -3) \cup (3, \infty)$ and concave down on $(-3, 3)$; No inflection points

1A/Archive/Homeworks - Spring 2011/hw10graph1.png



4.5.43.

D : $\mathbb{R}$
I : No $x-$intercepts, $y-$intercept: $y = \frac{1}{2}$
S : No symmetries
A : Horizontal Asymptotes: $y = 0$ (at $-\infty$), $y = 1$ (at ∞)
I : $f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$, so f is decreasing on $\mathbb{R}$
C : $f''(x) = \frac{e^x e^x - 1}{e^x + 1^3}$ (multiply numerator and denominator by $(e^x)^3$ after simplifying), so f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Inflection point at $(0, \frac{1}{2})$

1A/Math 1A Summer/Solution Bank/hw10graph3.png

4.5.45.

D : $x > 0$

I : No $x-$intercept because $f(x) > 0$ for all x (see Increasing/Decreasing section). No $y-$intercept (not defined at 0)

S : No symmetries

A : Vertical asymptote $x = 0$, No Horizontal Asymptote, because:

$$\lim_{x\to\infty} x - \ln(x) = \lim_{x\to\infty} x\left(1 - \frac{\ln(x)}{x}\right) = \infty(1-0) = \infty$$

Also no slant asymptote, because if there were such a slant asymptote $y = ax + b$, then:

$$a = \lim_{x\to\infty} \frac{\ln(x) - x}{x} = -1$$

And then:

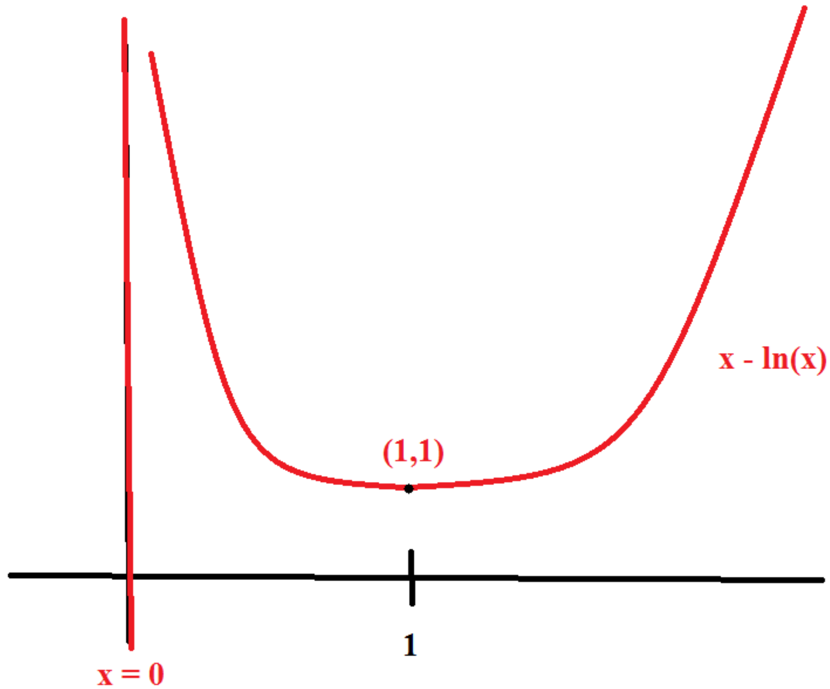
$$b = \lim_{x\to\infty} (\ln(x) - x) - (-1)x = \lim_{x\to\infty} \ln(x) = \infty$$

which is a contradiction!

I : $f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$, so decreasing on $(0,1)$ and increasing on $(1,\infty)$; local minimum $f(x) = 1$. In particular $f(x) \geq 1$ for all x, and so $f(x) > 0$ (hence no $x-$intercept)

C : $f''(x) = \frac{1}{x^2}$, concave up on $(0,\infty)$; No inflection points.

1A/Math 1A - Fall 2013/Homeworks/x - ln(x).png



4.5.49.

Note: : First of all, $\boxed{f \text{ is periodic of period } 2\pi}$, so from now on we may assume that $x \in [0, 2\pi]$

D : We want $\sin(x) > 0$, so the $\boxed{\text{domain is } (0, \pi)}$

I : $\boxed{\text{No } y\text{–intercepts}}$, x–intercepts: Want $\ln(\sin(x)) = 0$, so $\sin(x) = 1$, so $\boxed{x = \frac{\pi}{2}}$

S : Again, f is periodic of period 2π

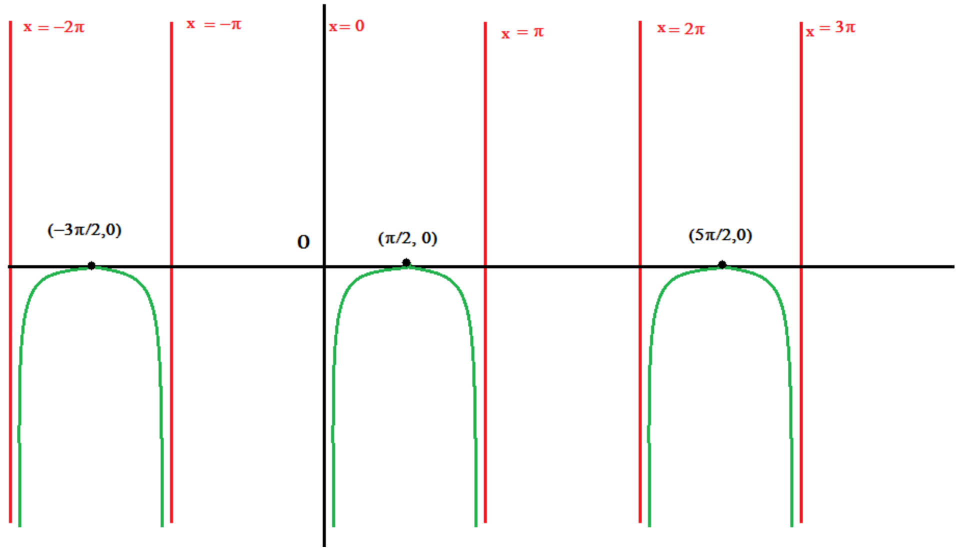
A : No horizontal/slant asymptotes, but $\lim_{x\to 0^+} \ln(\sin(x)) = \ln(0^+) = -\infty$, so $\boxed{x = 0}$ is a vertical asymptote. Also $\lim_{x\to\pi^-} \ln(\sin(x)) = -\infty$, so $\boxed{x = \pi}$ is also a vertical asymptote.

I : $f'(x) = \frac{\cos(x)}{\sin(x)} = \cot(x)$, then $f'(x) = 0 \Leftrightarrow x = \frac{\pi}{2}$, and using a sign table, we can see that $\boxed{f \text{ is increasing on } (0, \frac{\pi}{2}) \text{ and decreasing on } (\frac{\pi}{2}, \pi)}$.

Moreover, $\boxed{f(\frac{\pi}{2}) = \ln(1) = 0}$ is a local maximum of f.

C : $f''(x) = -\csc^2(x) < 0$, so $\boxed{f \text{ is concave down on } (0, \pi)}$.

1A/Math 1A Summer/Solution Bank/hw10graph.png



4.5.71. At ∞:

Suppose the slant asymptote is $y = ax + b$, then:

$$a = \lim_{x\to\infty} \frac{x - \tan^{-1}(x)}{x} = \lim_{x\to\infty} 1 - \frac{\tan^{-1}(x)}{x} = 1 - \frac{\frac{\pi}{2}}{\infty} = 1$$

$$b = \lim_{x\to\infty} x - \tan^{-1}(x) - x = \lim_{x\to\infty} -\tan^{-1}(x) = -\frac{\pi}{2}$$

Hence $x - \tan^{-1}(x)$ has a slant asymptote of $y = x - \frac{\pi}{2}$ at ∞

At $-\infty$:

Suppose the slant asymptote is $y = ax + b$, then:

$$a = \lim_{x\to-\infty} \frac{x - \tan^{-1}(x)}{x} = \lim_{x\to-\infty} 1 - \frac{\tan^{-1}(x)}{x} = 1 - \frac{\frac{\pi}{2}}{-\infty} = 1$$

$$b = \lim_{x\to-\infty} x - \tan^{-1}(x) - x = \lim_{x\to-\infty} -\tan^{-1}(x) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

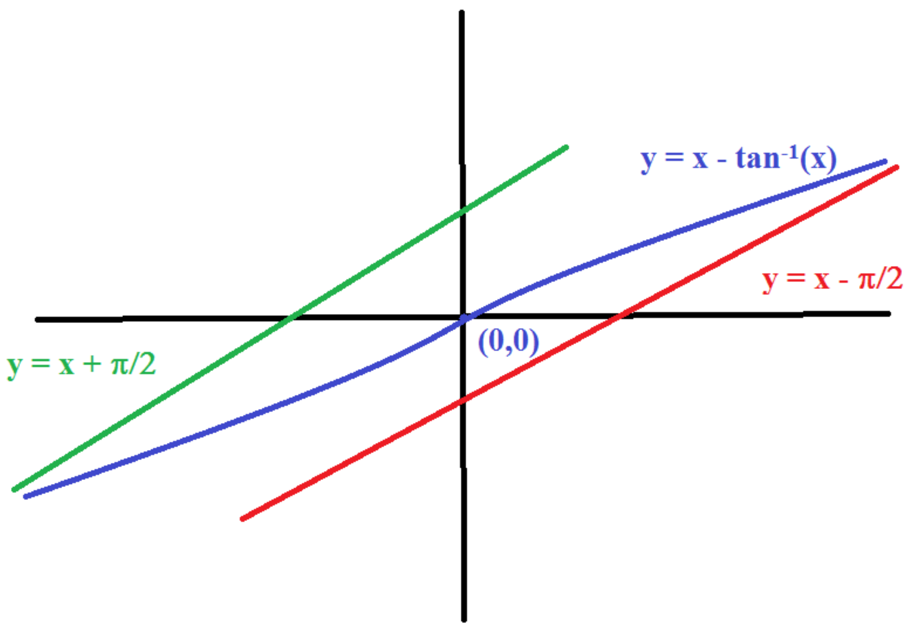
Hence $x - \tan^{-1}(x)$ has a slant asymptote of $y = x + \frac{\pi}{2}$ at $-\infty$

- D $Dom = \mathbb{R}$
- I $y-$intercept: $f(0) = 0$, $x-$intercept: 0 (there are no others, because f is increasing; see Increasing/Decreasing section)
- S No symmetries
- A No vertical asymptotes (f is defined everywhere), Slant Asymptotes $y = x - \frac{\pi}{2}$ at ∞, $y = x + \frac{\pi}{2}$ at $-\infty$; No H.A. because there are already two S.A.

I $f'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} \geq 0$, so f is increasing everywhere; No local max/min

C $f''(x) = \frac{2x\left(1+x^2\right)-x^2(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$, so f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Inflection point $(0, f(0)) = (0, 0)$

1A/Math 1A - Fall 2013/Homeworks/x - arctan(x).png



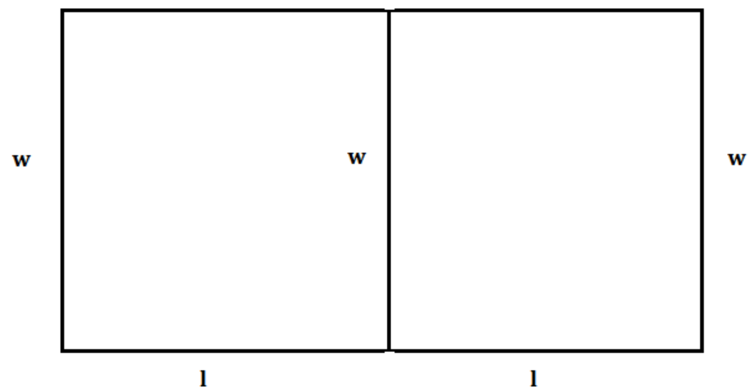
Section 4.7: Optimization Problems

4.7.3.

- Want to minimize $x + y$
- But $xy = 100$, so $y = \frac{100}{x}$, so $x + y = x + \frac{100}{x}$
- Let $f(x) = x + \frac{100}{x}$
- $x > 0$ (x is positive)
- $f'(x) = 0 \Leftrightarrow 1 - \frac{100}{x^2} = 0 \Leftrightarrow x^2 = 100 \Leftrightarrow x = 10$
- By FDTAEV, $x = 10$ is the absolute minimizer of f
- Answer: $\boxed{x = 10,\ y = \frac{100}{10} = 10}$

4.7.13. The picture is as follows:

1A/Math 1A Summer/Solution Bank/Fence.png



- Want to minimize $3w + 4l$
- But $2lw = 1.5$, so $l = \frac{0.75}{w}$, so $3w + 4l = 3w + \frac{3}{w}$
- Let $f(w) = 3w + \frac{3}{w}$
- $w > 0$
- $f'(x) = 0 \Leftrightarrow 3 - \frac{3}{w^2} = 0 \Leftrightarrow w^2 = 1 \Leftrightarrow w = 1$
- By FDTAEV, $w = 1$ is the absolute minimum of f
- Answer: $\boxed{w = 1,\ 2l = 1.5}$

4.7.14.

- Want to minimize $S = x^2 + 4xh$ (where x is the length of the base-side and h is the height)
- However, $V = x^2 h = 32000$, so $h = \frac{32000}{x^2}$, so $x^2 + 4xh = x^2 + 4x\frac{32000}{x^2} = x^2 + \frac{128000}{x}$
- Let $f(x) = x^2 + \frac{128000}{x}$
- $x > 0$
- $f'(x) = 0 \Leftrightarrow 2x - \frac{128000}{x^2} = 0 \Leftrightarrow 2x^3 = 128000 \Leftrightarrow x = \sqrt[3]{64000} = 40$
- By FDTAEV, $x = 40$ is the absolute minimizer of f
- Answer: $\boxed{x = 40,\ h = \frac{32000}{(40)^2} = \frac{32000}{1600} = 20}$

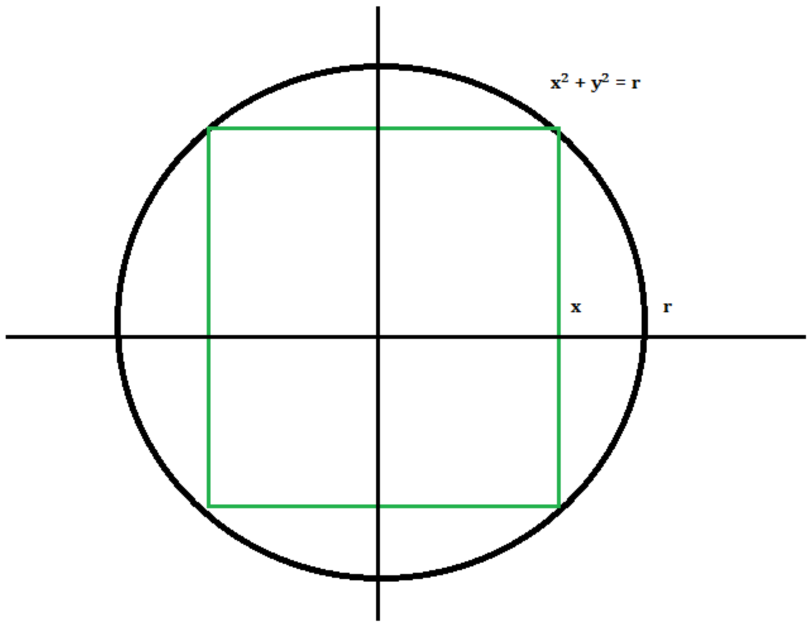
4.7.21.

- We have $D = \sqrt{(x-1)^2 + y^2}$, so $D^2 = (x-1)^2 + y^2$
- But $y^2 = 4 - 4x^2$, so $D^2 = (x-1)^2 + 4 - 4x^2$
- Let $f(x) = (x-1)^2 + 4 - 4x^2$
- No constraints
- $f'(x) = 2(x-1) - 8x = -6x - 2 = 0 \Leftrightarrow x = -\frac{1}{3}$
- By the FDTAEV, $x = -\frac{1}{3}$ is the maximizer of f.
- Since $y^2 = 4 - 4x^2$, we get $y^2 = 4 - \frac{4}{9} = \frac{32}{9}$, so $y = \pm\sqrt{\frac{32}{9}} = \pm\frac{4\sqrt{2}}{3}$

- Answer: $\boxed{\left(-\frac{1}{3}, -\frac{4\sqrt{2}}{3}\right) \text{ and } \left(-\frac{1}{3}, \frac{4\sqrt{2}}{3}\right)}$

4.7.23. Picture:

1A/Math 1A Summer/Solution Bank/hw10opt1.png



- We have $A = xy$, but the trick here again is to maximize $A^2 = x^2y^2$ (thanks for Huiling Pan for this suggestion!)
- But $x^2 + y^2 = r^2$, so $y^2 = r^2 - x^2$, so $A^2 = x^2(r^2 - x^2) = x^2r^2 - x^4$
- Let $f(x) = x^2r^2 - x^4$
- Constraint $0 \leq x \leq r$ (look at the picture)
- $f'(x) = 2xr^2 - 4x^3 = 0 \Leftrightarrow x = 0$ or $x = \frac{r}{\sqrt{2}}$
- By the closed interval method, $x = \frac{r}{\sqrt{2}}$ is a maximizer of f (basically $f(0) = f(r) = 0$
- Answer: $\boxed{x = \frac{r}{\sqrt{2}}, \ y = \sqrt{r^2 - \frac{r^2}{2}} = \frac{r}{\sqrt{2}}}$

4.7.32.

- $A = 2rh + \frac{1}{2}\pi r^2$
- But $P = \pi r + 2r + 2h = 30$, so $h = 15 - r - \frac{\pi}{2}r$
- Let $f(r) = 2r\left(15 - r - \frac{\pi}{2}r\right) + \frac{1}{2}\pi r^2 = 30r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2 = 30r - 2r^2 - \frac{\pi}{2}r^2$
- Constraint $r > 0$
- $f'(r) = 30 - 4r - \pi r = 0 \Leftrightarrow r = \frac{30}{\pi+4}$
- By FDTAEV, $r = \frac{30}{\pi+4}$ is the minimizer of f
- $\boxed{r = \frac{30}{\pi+4}, \ h = 15 - \frac{30}{\pi+4} - \frac{15\pi}{\pi+4} = \frac{30}{\pi+4} = r}$

4.7.48.

- Let t_{AB} be the time spent rowing from A to B and t_{BC} be the time spent walking from B to C
- By the formula time $= \frac{\text{distance}}{\text{velocity}}$, we have:

$$t_{AB} = \frac{AB}{2} = \frac{\cos(\theta)AC}{2} = \frac{4\cos(\theta)}{2} = 2\cos(\theta)$$

$$t_{BC} = \frac{BC}{4} = \frac{2 \times \angle BOC}{4} = \frac{2 \times 2\theta}{4} = \theta$$

(here O is the origin; it is a geometric fact that $\angle BOC = 2\angle BAC$)
- Let $f(\theta) = 2\cos(\theta) + \theta$
- Constraint: $0 \leq \theta \leq \frac{\pi}{2}$ (see the picture!)
- $f'(\theta) = -2\sin(\theta) + 1 = 0 \Leftrightarrow \sin(\theta) = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$
- $f(0) = 2$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ and $f\left(\frac{\pi}{3}\right) = 2 \times \frac{\sqrt{3}}{2} + \frac{\pi}{3} = \sqrt{3} + \frac{\pi}{3}$.

  By the **closed interval method**, $\theta = \frac{\pi}{2}$ is an absolute minimizer.
- Therefore, she should just walk! (which makes sense because she walks much faster than she rows!)

4.7.57. (a) $c'(x) = \frac{C'(x)x - C(x)}{x^2}$. When c is at its minimum, $c'(x) = 0$, so $C'(x)x - C(x) = 0$, so $C'(x) = \frac{C(x)}{x} = c(x)$, so $\boxed{C'(x) = c(x)}$, i.e. marginal cost equals the average cost!

4.7.61.

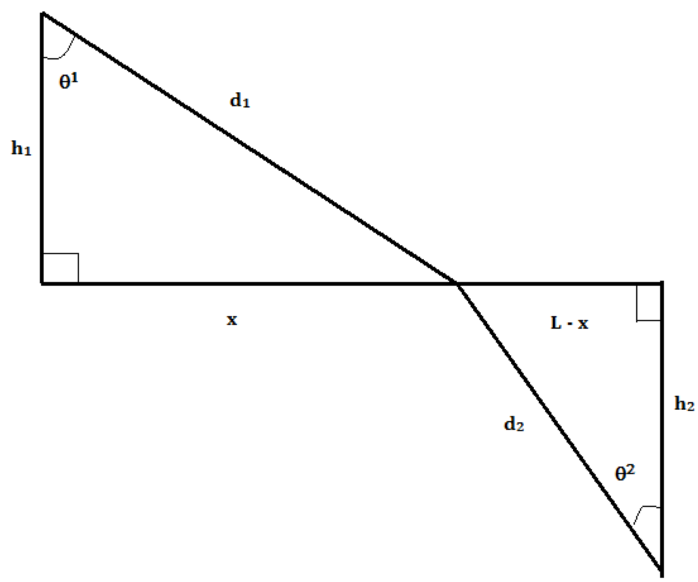
(a) $p(x) = 550 - \frac{x}{10}$ (Basically imitate Example 6 on page 331)

(b) The revenue function is $R(x) = xp(x) = 550x - \frac{x^2}{10}$. $R'(x) = 0 \Leftrightarrow 550 = \frac{x}{5} \Leftrightarrow x = 2750$, and the corresponding price is $p(2750) = 550 - 275 = 275$ and the rebate is $450 - 275 = 175$ dollars

(c) Here the profit function is $P(x) = R(x) - C(x) = 550x - \frac{x^2}{10} - 68000 - 150x$. $P'(x) = 0 \Leftrightarrow 550 - \frac{x}{5} - 150 = 0 \Leftrightarrow x = 2000$, so the corresponding price is $p(2000) = 550 - 200 = 350$, so the corresponding rebate is $450 - 350 = 100$ dollars

4.7.67. (thank you Brianna Grado-White for the solution to this problem!)
The picture is as follows:

1A/Math 1A Summer/Solution Bank/hw10opt2.png



Here, h_1 and h_2 and L are fixed, but x varies.
Now the total time taken is $t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2}$.

Now, by the Pythagorean theorem: $d_1 = \sqrt{x^2 + h_1^2}$ and $d_2 = \sqrt{(L-x)^2 + h_2^2}$, so we get:

$$t(x) = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(L-x)^2 + h_2^2}}{v_2}$$

And

$$t'(x) = \frac{x}{v_1\sqrt{x^2 + h_1^2}} + \frac{x - L}{v_2\sqrt{(L-x)^2 + h_2^2}} = \frac{x}{v_1 d_1} + \frac{x - L}{v_2 d_2}$$

Setting $t'(x) = 0$ and cross-multiplying, we get:

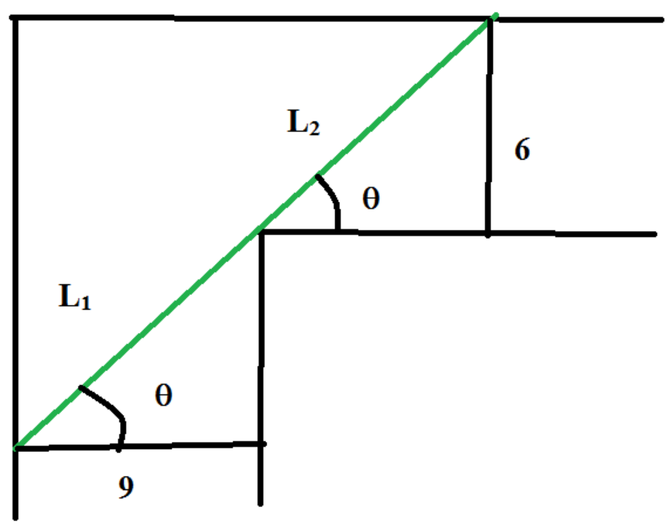
$$v_1 d_1 (L - x) = v_2 d_2 x$$

So, by definition of $\sin(\theta_1)$ and $\sin(\theta_2)$), we get:

$$\frac{v_1}{v_2} = \frac{d_2 x}{(L-x)d_1} = \frac{\frac{x}{d_1}}{\frac{L-x}{d_2}} = \frac{\sin(\theta_1)}{\sin(\theta_2)}$$

4.7.70. The picture is as follows (Note that the two $\theta-$s are indeed the same!)

1A/Math 1A - Fall 2013/Homeworks/Pipe.png



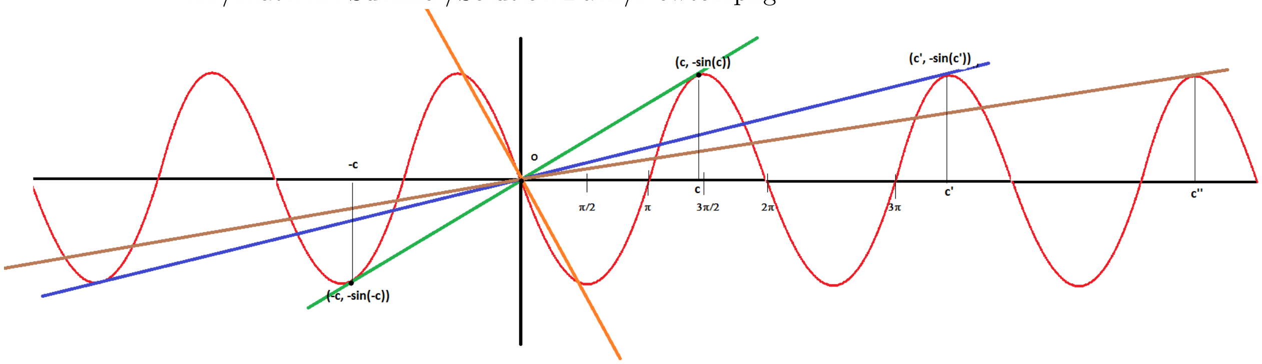
- We want to minimize $L_1 + L_2$
- $\cos(\theta) = \frac{L_1}{9}$, so $L_1 = \frac{9}{\cos(\theta)}$, $\sin(\theta) = \frac{L_2}{6}$, so $L_2 = \frac{6}{\sin(\theta)}$
- Let $f(\theta) = \frac{9}{\cos(\theta)} + \frac{6}{\sin(\theta)}$
- Constraint: $0 < \theta < \frac{\pi}{2}$ (Notice that at 0 and $\frac{\pi}{2}$, we can't carry the pipe horizontally around the corner; it would break at that corner)
- $f'(\theta) = \frac{9\sin(\theta)}{\cos^2(\theta)} + \frac{-6\cos(\theta)}{\sin^2(\theta)} = \frac{9\sin^3(\theta) - 6\cos^3(\theta)}{\cos^2(\theta)\sin^2(\theta)} = 0$

  $\Leftrightarrow 9\sin^3(\theta) - 6\cos^3(\theta) = 0 \Leftrightarrow \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^3 = \frac{6}{9} = \frac{2}{3} \Leftrightarrow \tan^3(\theta) = \frac{2}{3} \Leftrightarrow \theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$
- By FDTAEV, $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ is the absolute minimizer of f
- Answer: $\frac{9}{\cos(\theta)} + \frac{6}{\sin(\theta)}$, where $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ (if you want to, you can simplify this using the triangle method: $\frac{1}{\cos(\tan^{-1}(x))} = \sqrt{1+x^2}$ and $\frac{1}{\sin(\tan^{-1}(x))} = \frac{\sqrt{1+x^2}}{x}$, but I think this is enough torture for now :)

Section 4.8: Newton's method

4.8.38. As usual, a good picture is the key to solving the problem:

1A/Math 1A Summer/Solution Bank/Newton.png



Note: This picture is very complete. It is meant to illustrate all the points I am making below.

Here, we are given a lot of information, so let's try to tackle this problem one step at a time!

First, let's calculate the equation of *any* tangent line to the graph of $f(x) = -sin(x)$ that goes through $(0,0)$. Later, we will be worrying about finding the one that has the largest slope.

By definition of the derivative, the tangent line to the graph of f at c has slope $f'(c) = -\cos(c)$, so any such tangent line that **also** goes through $(0,0)$ has equation: $y - 0 = -\cos(c)(x-0)$, i.e. $y = -\cos(c)x$. Finally, we know that the tangent line goes through $(c, -\sin(c))$ (i.e. goes through the graph of f at c), so we get: $-\sin(c) = -\cos(c) \cdot c$, i.e. $\tan(c) = c$.

So any tangent line at c with the above properties must solve $\tan(c) = c$, i.e. $\tan(c) - c = 0$.

So what we really need to do is to approximate the zero of the function $g(x) = \tan(x) - x$. Now this looks like a Newton's method problem! But remember, that for Newton's method, we need to find a good initial guess, and **here** is where we use the information that the tangent line must have largest slope!

First of all, notice that $f(x) = -\sin(x)$ is odd, so the graph is symmetric about the origin! If you look at the picture above, you'll see that the tangent line to the graph at c is the same as the tangent line at $-c$. This means we can restrict ourselves to the right-hand-side of the picture, i.e. $c \geq 0$! (if you don't understand this argument, don't worry, it's just a simplification)

And if you look at the picture again, you'll notice that if your initial guess is between 0 and $\frac{\pi}{2}$, your successive approximations will got to 0. And you don't want that because the slope of the tangent line at 0 is -1 (which is not the greatest slope). The same problem arises with the initial guess between $\frac{3\pi}{2}$ and 2π (the approximations go to 2π)

Finally, notice that when c gets larger and larger, the tangent line at c has smaller and smaller slope (see picture: The brown line has a smaller slope than the blue line, which has a smaller slope than the green line), so you'd like your initial guess not to be too large. In particular, we don't want the initial guess to be larger than $\frac{3\pi}{2}$!

From this analysis, we conclude that any initial guess between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ is good enough! (also look at the picture above: the green tangent line seems to be the winner here)

For example, with $x_0 = 4.5$ (or you could try $x_0 = \pi$), we get the successive approximations (remember that you are applying Newton's method to $g(x) = \tan(x) - x$, **NOT** $f(x)$!):

$$\begin{aligned} x_0 &= 4.5 \\ x_1 &= 4.49361390 \\ x_2 &= 4.49340966 \\ x_3 &= 4.49340946 \\ x_4 &= 4.49340946 \end{aligned}$$

And so, our approximation is: $\boxed{c \approx 4.49340946}$. And hence the largest **slope** is approximately equal to $\boxed{-\cos(4.49340946) \approx 0.2172336}$ (because $f'(c) = -\cos(c)$).

<u>To summarize</u>:

- Draw a picture
- Derive the function that you want to apply Newton's method to (i.e. $g(x) = \tan(x) - x$)
- Argue that your intial approximation must be between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ (**YOU NEED TO JUSTIFY THIS PART!**, maybe not as precise as I did, but there needs to be some justification
- Apply Newton's method to g with initial approximation between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ (4 would work)

Section 4.9: Antiderivatives

4.9.16. $R(\theta) = \sec(\theta) - 2e^{\theta}$

4.9.35. $f(x) = -2\sin(t) + \tan(t) + C$, but $4 = f(\frac{\pi}{3}) = -\sqrt{3} + \sqrt{3} + C = C$, so $f(x) = -2\sin(t) + \tan(t) + 4$

4.9.41. If $f''(\theta) = \sin(\theta) + \cos(\theta)$, then $f'(\theta) = -\cos(\theta) + \sin(\theta) + C$.
$f'(0) = 4$, so $-1 + 0 + C = 4$, so $C = 5$.
Hence $f'(\theta) = -\cos(\theta) + \sin(\theta) + 5$.
Hence $f(\theta) = -\sin(\theta) - \cos(\theta) + 5\theta + C'$.

$f(0)=3$, so $-0-1+0+C'=3$, so $C'=4$.
Hence $\boxed{f(\theta)=-\sin(\theta)-\cos(\theta)+5\theta+4}$

4.9.63. $a(t)=10\sin(t)+3\cos(t)$, so $v(t)=-10\cos(t)+3\sin(t)+A$, so $s(t)=-10\sin(t)-3\cos(t)+At+B$

Now, $s(0)=0$, but $s(0)=-10(0)-3(1)+A(0)+B$, so $-3+B=0$, so $B=3$

So $s(t)=-10\sin(t)-3\cos(t)+At+3$

Moreover, $s(2\pi)=12$, but $s(2\pi)=-10(0)-3(1)+A(2\pi)+3=A(2\pi)$, so $A(2\pi)=12$, so $A=\frac{12}{2\pi}=\frac{6}{\pi}$

So altogether, you get: $\boxed{s(t)=-10\sin(t)-3\cos(t)+\frac{6}{\pi}t+3}$

4.9.76. Suppose the acceleration of the car is $a(t)=A$. Then $v(t)=At+B$ and $s(t)=\frac{A}{2}t^2+Bt+C$.

However, at $t=0$, the car is moving at 100 km/h, so $v(0)=100$, so $B=100$, hence $v(t)=At+100$ and $s(t)=\frac{A}{2}t^2+100t+C$.

Moreover, at $t=0$, the car is at its initial position 0, so $s(0)=0$, so $C=0$, hence $s(t)=\frac{A}{2}t^2+100t$

Now let t^* be the time needed to real the pile-up.

We want the car to have 0 velocity at t^*, hence $v(t^*)=0$, hence $At^*+100=0$, so $At^*=-100$

Moreover, we want $s(t^*)=80\text{m}=0.08$ km, so $\frac{A}{2}\left(t^*\right)^2+100t^*=0.08$, but using the fact that $At^*=-100$, this just becomes: $\frac{-100t^*}{2}+100t^*=0.08$, so $50t^*=0.08$, so $t^*=\frac{1}{625}$.

Therefore $A=-\frac{100}{t^*}=-100\times 625=-62500\ km/h^2$, so the answer is $\boxed{62500\ km/h^2}$.

Section 5.1: Areas and Distances

5.1.2.
(a) (i) $\Delta x=2$, so

$$L_6=f(0)(2)+f(2)(2)+f(4)(2)+f(6)(2)+f(8)(2)+f(10)(2)=18+\frac{52}{3}+\frac{50}{3}+\frac{44}{3}+12+8=\frac{260}{3}\approx 86.67$$

(ii)

$$R_6=f(2)(2)+f(4)(2)+f(6)(2)+f(8)(2)+f(10)(2)+f(12)(2)=\frac{52}{3}+\frac{50}{3}+\frac{44}{3}+12+8+2=\frac{212}{3}\approx 70.67$$

(iii)

$$M_6=f(1)(2)+f(3)(2)+f(5)(2)+f(7)(2)+f(9)(2)+f(11)(2)=18+17+15+13+10+\frac{16}{3}=\frac{235}{3}\approx 78.33$$

(b) Overestimate
(c) Underestimate
(d) M_6 (just right, does not overshoot, like L_6, but not undershoot either, like R_6)

5.1.5.

(a) If $n = 3$, then $\Delta x = 1$, and if $n = 6$, $\Delta x = \frac{1}{2}$, so:

$$R_3 = f(0)(1) + f(1)(1) + f(2)(1) = 1 + 2 + 5 = 8$$

$$\begin{aligned} R_6 =& f(-0.5)(0.5) + f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) \\ =& 1.25(0.5) + 1(0.5) + 1.25(0.5) + 2(0.5) + 3.25(0.5) + 5(0.5) \\ =& 6.875 \end{aligned}$$

(b)

$$L_3 = f(-1)(1) + f(0)(1) + f(1)(1) = 2 + 1 + 2 = 5$$

$$\begin{aligned} L_6 =& f(-1)(0.5) + f(-0.5)(0.5) + f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) \\ =& 2(0.5) + 1.25(0.5) + 1(0.5) + 1.25(0.5) + 2(0.5) + 3.25(0.5) \\ =& 5.375 \end{aligned}$$

(c)

$$M_3 = f(-0.5)(1) + f(0.5)(1) + f(1.5)(1) = 5.75$$

$$M_6 = f(-0.75)(0.5)+f(-0.25)(0.5)+f(0.25)(0.5)+f(0.75)(0.5)+f(1.25)(0.5)+f(1.75)(0.5) = 5.9375$$

(d) M_6

5.1.13. Here $n = 6$ and $\Delta x = 0.5$

$$\begin{aligned} L_6 =& v(0)(0.5) + v(0.5)(0.5) + v(1)(0.5) + v(1.5)(0.5) + v(2)(0.5) + v(2.5)(0.5) \\ =& 0(0.5) + 6.2(0.5) + 10.8(0.5) + 14.9(0.5) + 18.1(0.5) + 19.4(0.5) \\ =& 34.7 \end{aligned}$$

$$\begin{aligned} R_6 =& v(0.5)(0.5) + v(1)(0.5) + v(1.5)(0.5) + v(2)(0.5) + v(2.5)(0.5) + v(3)(0.5) \\ =& 6.2(0.5) + 10.8(0.5) + 14.9(0.5) + 18.1(0.5) + 19.4(0.5) + 20.2(0.5) \\ =& 44.8 \end{aligned}$$

5.1.17. The midpoint sum seems to best approximate the area:

$$M_6 = v(0.5)(1)+v(1.5)(1)+v(2.5)(1)+v(3.5)(1)+v(4.5)(1)+v(5.5)(1) = 50+40+30+18+10+5 = 153ft$$

5.1.19. Here $\Delta x = \frac{2}{n}$ and $x_i = 1 + \frac{2i}{n}$, so:

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{2}{n} \frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2 + 1}$$

5.1.20. Here $\Delta x = \frac{\pi}{n}$ and $x_i = \frac{\pi i}{n}$, so:

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{\pi}{n} \sqrt{\sin\left(\frac{\pi i}{n}\right)}$$

5.1.22. The area under the curve of $f(x) = x^{10}$ from 5 to 7 (or, if you want, the area under the curve of $f(x) = (x+5)^{10}$ from 0 to 2)

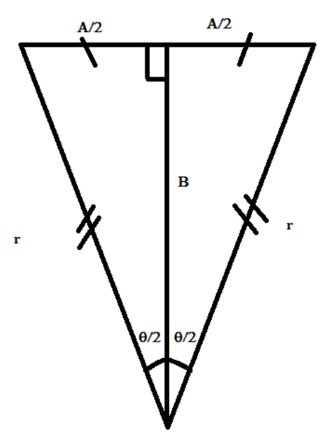
5.1.23. The area under the curve of $f(x) = \tan(x)$ from 0 to $\frac{\pi}{4}$

5.1.30.

(a) We are given that the polygon is made out of n congruent triangles, so $A_n = n \cdot T$, where T is the area of each triangle. So all we need to find is T.

Here again, a picture tells a thousand words, so by drawing the picture of such a triangle, we can figure out its area:

1A/Math 1A Summer/Solution Bank/Polygon.png



Using the picture, you'll notice that:

$$T = \frac{1}{2} \cdot A \cdot B = \frac{A}{2} B$$

And we can divide the triangle into two right triangles, and hence use trigonometry to calculate $\frac{A}{2}$ and B! Here, $\theta = \frac{2\pi}{n}$, the central angle!

We get:

$$\cos\left(\frac{\theta}{2}\right) = \frac{B}{r}$$
$$B = r \cdot \cos\left(\frac{\theta}{2}\right)$$

$$\sin\left(\frac{\theta}{2}\right) = \frac{\frac{A}{2}}{r}$$
$$\frac{A}{2} = r \cdot \sin\left(\frac{\theta}{2}\right)$$

And so, we get:

$$T = \frac{A}{2} \cdot B = r \cdot \sin\left(\frac{\theta}{2}\right) \cdot r \cdot \cos\left(\frac{\theta}{2}\right) = r^2 \cdot \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = r^2 \frac{1}{2}\sin\left(2 \cdot \frac{\theta}{2}\right) = \frac{1}{2}r^2\sin(\theta) = \frac{1}{2}r^2\sin\left(\frac{2\pi}{n}\right)$$

Here, we used the fact that, in general, $2\sin(x)\cos(x) = \sin(2x)$, so $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$.

And so, we have:

$$A_n = n \cdot T = n \cdot \frac{1}{2}r^2\sin\left(\frac{2\pi}{n}\right) = \frac{1}{2}nr^2\sin\left(\frac{2\pi}{n}\right)$$

(b) Actually, the hint tells us that we don't even have to use l'Hopital's rule, but rather the rule that:

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

We have that:

$$\begin{aligned}
\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{1}{2} r^2 n \sin\left(\frac{2\pi}{n}\right) \\
=& \frac{1}{2} r^2 \lim_{n\to\infty} n \sin\left(\frac{2\pi}{n}\right) \\
=& \frac{1}{2} r^2 \lim_{n\to\infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}} \\
=& \frac{1}{2} r^2 \lim_{n\to\infty} \frac{2\pi \sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \\
=& 2\pi \cdot \frac{1}{2} r^2 \lim_{n\to\infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \\
=& \pi r^2 \lim_{x\to 0} \frac{\sin(x)}{x} && \left(x = \frac{2\pi}{n}\right) \\
=& \pi r^2 (1) \\
=& \pi r^2
\end{aligned}$$

The basic idea is that, if we have an indeterminate form of the form "$0 \cdot \infty$", we rewrite ∞ as $\frac{1}{0}$, or we rewrite 0 as $\frac{1}{\infty}$. Here, for example, we wrote $n = \frac{1}{\frac{1}{n}}$ in order to apply the hint in the problem!

And hooray, you just proved that the formula for the area of a circle of radius r is πr^2. But actually, you didn't, because trigonometry, which you used in (a), relies heavily on this formula!

Section 5.2: The definite integral

5.2.18. $\int_{\pi}^{2\pi} \frac{\cos(x)}{x} dx$

5.2.21. First of all, $a = 2$, $b = 5$, $\Delta x = \frac{3}{n}$ and $x_i = 2 + \frac{3i}{n}$

$$\begin{aligned}
\int_2^5 4-2xdx &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(4-2\left(2+\frac{3i}{n}\right)\right)\\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(4-4-\frac{6i}{n}\right)\\
&= \lim_{n\to\infty}\frac{3}{n}\left(\sum_{i=1}^{n}\frac{-6}{n}i\right)\\
&= \lim_{n\to\infty}\frac{-18}{n^2}\left(\sum_{i=1}^{n}i\right)\\
&= \lim_{n\to\infty}\frac{-18}{n^2}\left(\frac{n(n+1)}{2}\right)\\
&= \lim_{n\to\infty}\frac{-9n^2}{n^2+n}\\
&= \lim_{n\to\infty}\frac{-9n^2}{n^2\left(1+\frac{1}{n}\right)}\\
&= \lim_{n\to\infty}\frac{-9}{\left(1+\frac{1}{n}\right)}\\
&= -9
\end{aligned}$$

5.2.23. First of all, $a=-2$, $b=0$, $\Delta x=\frac{2}{n}$ and $x_i=-2+\frac{2i}{n}$

$$\begin{aligned}
\int_{-2}^{0} x^2 + x dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(-2+\frac{2i}{n}\right)^2 + \left(-2+\frac{2i}{n}\right)\\
&= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(4-\frac{8i}{n}+\frac{4i^2}{n^2}-2+\frac{2i}{n}\right)\\
&= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}-\frac{6i}{n}+2\right)\\
&= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right)-\frac{2}{n}\sum_{i=1}^{n}\left(\frac{6i}{n}\right)+\frac{2}{n}\sum_{i=1}^{n}(2)\\
&= \lim_{n\to\infty} \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2\right)-\frac{12}{n^2}\left(\sum_{i=1}^{n} i\right)+\frac{4}{n}\left(\sum_{i=1}^{n} 1\right)\\
&= \lim_{n\to\infty} \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)-\frac{12}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{4}{n}n\\
&= \lim_{n\to\infty} \frac{8}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-6\left(1+\frac{1}{n}\right)+4\\
&=\frac{4}{3}\times 2-6+4\\
&=\frac{8}{3}-2\\
&=\frac{2}{3}
\end{aligned}$$

5.2.30. $\Delta x = \frac{9}{n}$, so $x_i = 1 + \frac{9i}{n}$, and so:

$$\int_{1}^{10} x - 4\ln(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta(x)(x_i - 4\ln(x_i)) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{9}{n}\left(\left(1+\frac{9i}{n}\right)-4\ln\left(1+\frac{9i}{n}\right)\right)$$

5.2.34.

(a) 4 (the area of the large triangle)
(b) -2π (minus the area of the semicircle)
(c) $4 - 2\pi + \frac{1}{2} = \frac{9}{2} - 2\pi$ (the area of the large triangle minus the area of the semicircle plus the area of the small triangle)

5.2.37. $3 + \frac{9\pi}{4}$ (the area of a rectangle plus the area of a quarter of a circle of radius 3)

5.2.43. $\int_0^1 5 - 6x^2 dx = 5\int_0^1 1dx - 6\int_0^1 x^2 dx = 5 - 6\frac{1}{3} = 5 - 2 = 3$

5.2.44. $\int_1^3 2e^x - 1dx = 2\int_1^3 e^x dx - \int_1^3 1dx = 2(e^3 - e) - 2$

5.2.47.

$$\int_{-2}^{2} f(x)dx+\int_{2}^{5} f(x)dx-\int_{-2}^{-1} f(x)dx=\int_{-2}^{5} f(x)dx-\int_{-2}^{-1} f(x)dx=\int_{-2}^{5} f(x)dx+\int_{-1}^{-2} f(x)dx=\int_{-1}^{5} f(x)dx$$

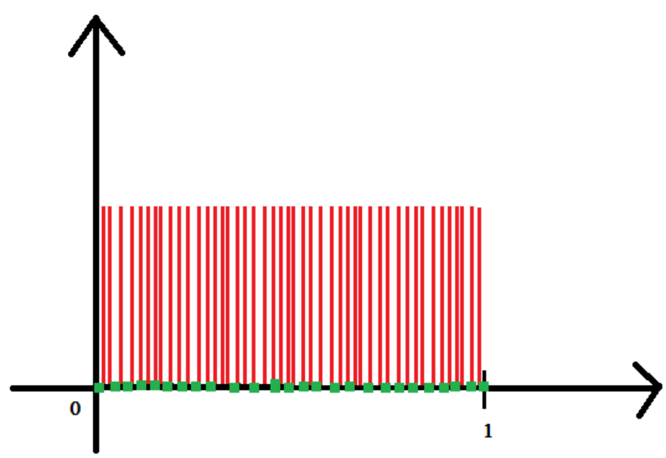
5.2.54. $m \leq f(x) \leq M$, so integrating from 0 to 2, we get $2m \leq \int_0^2 f(x)dx \leq 2M$

5.2.56. On $[0,1]$, $x^2 \leq x$, so $1+x^2 \leq 1+x$, so $\sqrt{1+x^2} \leq \sqrt{1+x}$, so integrating from 0 to 1, we get $\int_0^1 \sqrt{1+x^2}dx \leq \int_0^1 \sqrt{1+x}dx$

5.2.69. As usual, let $f(x)$ be as in the problem, $a=0$, $b=1$, $x_i=\frac{i}{n}$, and $\Delta(x)=\frac{1}{n}$.

First, let's draw a picture of what's going on:

1A/Math 1A Summer/Solution Bank/Nonintegrable.png



In the picture above, the green dots represent where $f(x)=0$ and the red lines represent where $f(x)=1$. This problem is unlike the problem above! In this case, the function does not blow up to infinity, but it can't make up its mind! We need to somehow use this fact in order to show that f is not integrable!

But even though this problem is different, the general strategy is almost the same. f integrable means that no matter how we choose the x_i^*, we get the same answer! So to show that something is **NOT** integrable, we have to pick two different sets of points x_i^* and y_i^* that give us two different answers!

And here is where we use the fact that f looks the way it does. Namely, let x_i^* be your favorite rational number in $[x_{i-1}, x_i]$ and y_i^* your favorite irrational number in $[x_{i-1}, x_i]$! For example (you don't have to write this, but it's better if you do!), you can choose:

$$x_i^* = x_i = \frac{1}{n}$$
$$y_i^* = \frac{i}{\sqrt{2}n}$$

And you can check that $x_i^* \in [x_{i-1}, x_i]$, and $y_i^* \in [x_{i-1}, x_i]$.

But the point is that x_i^* is rational, and so $f(x_i^*) = 0$ by definition of f, and thus the Riemann sum equals to:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta(x) = \lim_{n\to\infty} \sum_{i=1}^{n} 0 \cdot \frac{1}{n} = 0$$

And y_i^* is rational, and so $f(y_i^*) = 1$ by definition of f, and thus the Riemann sum equals to:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(y_i^*)\,\Delta(x) = \lim_{n\to\infty} \sum_{i=1}^{n} 1 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{n}{n} = \lim_{n\to\infty} 1 = 1$$

And so we get two different answers (even though we're supposed to get the same answer if f were integrable)!!! Which shows that f is not integrable on $[0, 1]$!

5.2.70. Let $f(x) = \frac{1}{x}$, $a = 0$, $b = 1$. Then $x_i = \frac{i}{n}$ and $\Delta(x) = \frac{1}{n}$.

How can we show that something is not integrable? The main point is: **Given n we need to CHOOSE a set of points $x_i^* \in [x_{i-1}, x_i]$ that 'fails'** (whatever that might mean). As discussed in section, the following choice is a good one:

$$x_1^* = \frac{1}{n^2}$$
$$x_i^* = x_i \qquad \text{for } i \geq 2$$

This works **BECAUSE** $x_1^* \in [x_0, x_1] = [0, \frac{1}{n}]$ and $x_i^* \in [x_{i-1}, x_i]$ (for $i \geq 2$). **Always check this on the exam!**

Because then, we have:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta(x) \geq f(x_1^*)\,\Delta(x) = \frac{1}{\frac{1}{n^2}} \cdot \frac{1}{n} = \frac{n^2}{n} = n$$

Here, we use the fact that every term in the sum is positive, so the sum is greater than its first term $f(x_1^*)\,\Delta(x)$. Also, $\Delta(x) = \frac{1}{n}$.

And now, if we let $n \to \infty$, the right-hand-side goes to ∞, and so by comparison,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta(x) = \infty$$

So with this choice of x_i^*, things have gone awry! The Riemann sum 'blows' up to infinity, and so f is not integrable over $[0,1]$. The point is: **if a function is integrable, then its integral has to be finite.**

Other solution:
Some people wrote up another solution, which is also pretty clever!

Basically, let $x_i^* = x_i = \frac{i}{n}$ $(1 \le i \le n)$, which is in $[x_{i-1}, x_i]$.

Then:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta(x) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n} = \sum_{i=1}^{n} \left(\frac{n}{i}\right) \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$$

However, some of you might know that:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} = \infty$$

It's ok if you don't know this, you're not even supposed to know this because it's covered in Math 1B! (that's why I don't know how many points you would actually get on the exam for this answer...)

And thus:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} = \infty$$

And hence f is not integrable on $[0,1]$ **WARNING:** Note that you **CANNOT**

just say that f is not integrable because it has a vertical asymptote at $x = 0$! For example, the function $g(x) = \frac{1}{\sqrt{x}}$ has a vertical asymptote at $x = 0$, but:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x}\right]_0^1 = 2$$

Because $2\sqrt{x}$ is an antiderivative of $\frac{1}{\sqrt{x}}$

Section 5.3: The Fundamental Theorem of Calculus

5.3.7. $\frac{1}{x^3+1}$

5.3.15. $\sec^2(x)\sqrt{\tan(x) + \sqrt{\tan(x)}}$

5.3.17. $3\frac{(1-3x)^3}{1+(1-3x)^2}$

5.3.27. $-\frac{37}{6}$ (Write $(u+2)(u-3) = u^2 - u - 6$)

5.3.31. 1 (antiderivative is $\tan(t)$)

5.3.35. $\ln(2)+7$ (Write $\frac{v^3+3v^6}{v^4} = \frac{1}{v} + \frac{3}{v^2}$, whose antiderivative is $\ln|v| - \frac{3}{v}$)

5.3.37. $\frac{1}{e+1} + (e-1)$ (Antiderivative is $\frac{x^{e+1}}{e+1} + e^x$)

5.3.39. $8\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{4\pi}{3}$ (antiderivative is $8\tan^{-1}(t)$)

5.3.43. $1 + (-1) = 0$ (split up the integral into $\int_0^{\frac{\pi}{2}} \sin(x)dx + \int_{\frac{\pi}{2}}^{\pi} \cos(x)dx$)

5.3.45. $\frac{1}{x^4}$ is discontinuous at 0 (the FTC applies only to continuous functions)

5.3.57. $F'(x) = 2xe^{x^4} - e^{x^2}$

5.3.67.

(a) $g'(x) = f(x) = 0 \Rightarrow x = 1, 3, 5, 7, 9$, but 9 is an endpoints, so ignore it. Hence, by the second derivative test:

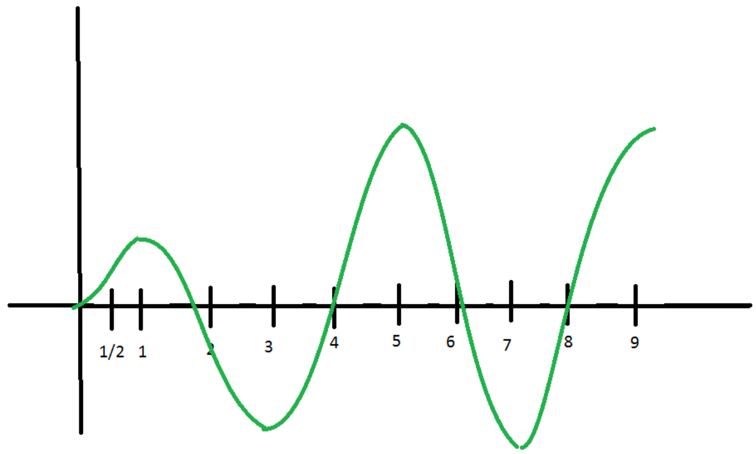
- $g''(1) = f'(1) < 0$, so g has a local max at 1
- $g''(3) = f'(3) > 0$, so g has a local min at 3
- $g''(5) = f'(5) < 0$, so g has a local max at 5
- $g''(7) = f'(7) > 0$, so g has a local min at 7

In summary, g attains a local minimum at $\boxed{\text{3 and 7}}$, and a local maximum at $\boxed{\text{1 and 5}}$.

(b) You do this by guessing. The candidates are $0, 1, 3, 5, 7, 9$ (critical points and endpoints). Notice $g(0) = 0, g(3) < 0$ but $g(5) > 0$, so you can eliminate 0 and 3. Also $g(5) > g(1)$, so you can eliminate 1. Also $g(7) < 0$, so you can eliminate 7. This leaves us with 5 and 9, but notice that $g(5) = g(9)$ (the areas between 5 and 9 cancel out), so the answer is $\boxed{x = 5 \text{ and } x = 9}$ (the book only writes $x = 9$, but I disagree)

(c) $g''(x) = f'(x)$, so to see where g is concave down, we have to check where $f'(x) < 0$, i.e. where f is decreasing. The answer is $\boxed{\left(\frac{1}{2}, 2\right) \cup (4, 6) \cup (8, 9)}$.

(d) 1A/Math 1A - Fall 2013/Homeworks/FTCSol.png



5.3.70. First rewrite the limit as:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}}$$

And you should recognize that $\Delta x = \frac{1}{n}$, $f(x) = \sqrt{x}$, $x_i = \frac{i}{n}$. In particular $a = x_0 = 0$ and $b = x_n = \frac{n}{n} = 1$, so in fact this limit equals to:

$$\int_0^1 \sqrt{x}dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$

Section 5.4: Indefinite Integrals and the Net Change Theorem

5.4.10. $\frac{1}{6}v^6 + v^4 + 2v^2 + C$ (expand out)

5.4.12. $\frac{x^3}{3} + x + \tan^{-1}(x) + C$

5.4.13. $-\cos(x) + \cosh(x) + C$

5.4.25. -2 (expand out)

5.4.31. $\frac{55}{63}$ (Write this as $x^{\frac{4}{3}} + x^{\frac{5}{4}}$, with antiderivative $\frac{3}{7}x^{\frac{7}{3}} + \frac{4}{9}x^{\frac{9}{4}}$)

5.4.37. $1 + \frac{\pi}{4}$ (Antiderivative is $\tan(\theta) + \theta$, because:)

$$\frac{1+\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \sec^2(\theta) + 1$$

5.4.49. $\frac{4}{3}$ (antiderivative is $y^2 - \frac{y^3}{3}$)

5.4.54. The bee population after 15 weeks

5.4.61.

(a) $v(t) = \frac{t^2}{2} + 4t + 5$

(b) $s(10) - s(0) = \frac{2500}{6}$ (antiderivative is $\frac{t^3}{6} + 2t^2 + 5t$)

5.4.62.

(a)

$$s(6)-s(1) = \int_1^6 v(t)dt = \int_1^6 \left(t^2 - 2t - 8\right) dt = \left[\frac{t^3}{3} - t^2 - 8t\right]_1^6 = -12+\frac{26}{3} = -\frac{10}{3}$$

(Alternatively, you could have just calculated $s(t)$ by antidifferentiating v and then calculated $s(6) - s(1)$ directly)

(b) Notice that $v(t) = (t+2)(t-4) = 0$, which gives $t = 4$ (since $t \geq 0$). So in particular $v(t) \leq 0$ on $[1, 4]$ (the particle is moving to the left) and $v(t) \geq 0$ on $[4, 6]$ (the particle is moving to the right), hence we must find:

$$\begin{aligned}
s(1) - s(4) + s(6) - s(4) &= -\left(s(4) - s(1)\right) + \left(s(6) - s(4)\right) \\
&= -\int_1^4 v(t)dt + \int_2^6 v(t)dt \\
&= -\int_1^4 \left(t^2 - 2t - 8\right) dt + \int_4^6 \left(t^2 - 2t - 8\right) dt \\
&= -\left[\frac{t^3}{3} - t^2 - 8t\right]_1^4 + \left[\frac{t^3}{3} - t^2 - 8t\right]_4^6 \\
&= -(-18) + \frac{44}{3} \\
&= \frac{98}{3}
\end{aligned}$$

5.4.63. $\frac{140}{3}$ (antiderivative is $9x + \frac{4}{3}x^{\frac{3}{2}}$, and $a = 0$, $b = 4$)

5.4.64. 1800 (antiderivative is $200t - 2t^2$, $a = 0$, $b = 10$)

Section 5.5: The substitution rule

5.5.7. $\frac{1}{2}\cos(x^2)$ ($u = x^2$, $du = 2xdx$)

5.5.31. $e^{\tan(x)} + C$ ($u = \tan(x)$, $du = \sec^2(x)dx$)

5.5.33. $-\frac{1}{\sin(x)}$ ($u = \sin(x)$, $du = \cos(x)dx$)

5.5.48. $\frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}}$ ($u = x^2 + 1$, $du = 2xdx$, $x^2 = u - 1$)

5.5.59. $e - \sqrt{e}$ ($u = \frac{1}{x}$, $du = -\frac{1}{x^2}dx$, $a = 1$, $b = \frac{1}{2}$)

5.5.62. $\sin(1)$ ($u = \sin(x)$, $du = \cos(x)$, $a = 0$, $b = 1$)

5.5.77. $0 + 6\pi$ (the first integral is 0 because the function is an odd function, or use $u = 4 - x^2$, $du = -2xdx$, $a = 0$, $b = 0$, and the second integral represents the area of a semicircle with radius 2)

5.5.86. Using the substitution $u = x^2$, we get $du = 2xdx$, so $xdx = \frac{1}{2}du$. Moreover, the endpoints become $u(0) = 0$ and $u(3) = 9$, so:

$$\int_0^3 xf\left(x^2\right)dx = \int_0^9 f(u)\frac{1}{2}du = \frac{1}{2}\int_0^9 f(x)dx = \frac{4}{2} = 2$$

5.5.92.

(a) For the first integral, let $u = \cos(x)$, then $du = -\sin(x)dx = -\sqrt{1-u^2}dx$, so the first integral becomes $\int_1^0 \frac{f(u)}{-\sqrt{1-u^2}}du = \int_0^1 \frac{f(u)}{\sqrt{1-u^2}}du$. For the second integral, let $u = \sin(x)$, then $du = \cos(x)dx = \sqrt{1-u^2}dx$, so the second integral becomes $\int_0^1 \frac{f(u)}{\sqrt{1-u^2}}du$, and it is now clear that both integrals are equal!

Note: Another solution is to use $u = \frac{\pi}{2} - x$ and use the fact that $\sin(x) = \sin\left(\frac{\pi}{2} - u\right) = \cos(u)$.

(b) By (a) with $f(x) = x^2$ (for the first step), and the fact that $\sin^2(x) = 1 - \cos^2(x)$, we get:

$$\int_0^{\frac{\pi}{2}} \cos^2(x)dx = \int_0^{\frac{\pi}{2}} \sin^2(x)dx = \int_0^{\frac{\pi}{2}} 1dx - \int_0^{\frac{\pi}{2}} \cos^2(x)dx = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \cos^2(x)dx$$

Solving for $\int_0^2 \cos^2(x)dx$, we get: $\boxed{\int_0^2 \cos^2(x)dx = \frac{\pi}{4}}$, and hence $\boxed{\int_0^2 \sin^2(x)dx = \frac{\pi}{4}}$ (by (a))

Section 6.1: Areas between curves

6.1.1. $\int_0^4 (5x - x^2) - xdx = \int_0^4 4x - x^2dx = \boxed{\frac{32}{3}}$

6.1.3. $\int_{-1}^1 e^y - (y^2 - 2)dy = \boxed{e - e^{-1} + \frac{10}{3}}$

6.1.13. $\int_{-3}^3 (12 - x^2) - (x^2 - 6)dx = \int_{-3}^3 18 - 2x^2dx = = \boxed{72}$
(points of intersection are $x = \pm 3$)

6.1.21. To find the points of intersection, solve:

$$\begin{aligned}\tan(x) =&2\sin(x)\\ \frac{\sin(x)}{\cos(x)} =&2\sin(x)\\ \sin(x) =&2\sin(x)\cos(x)\\ \sin(x)(1-2\cos(x)) =&0\end{aligned}$$

which implies either $\sin(x)=0$, that is $x=0$, or $\cos(x)=\frac{1}{2}$, that is, $x=\pm\frac{\pi}{3}$.

Hence, if you draw a good picture, you'll see that we need to find:

$$\int_{-\frac{\pi}{3}}^{0} \tan(x)-2\sin(x)dx + \int_{0}^{\frac{\pi}{3}} 2\sin(x)-\tan(x)dx$$

But by symmetry (see your picture), both of those integrals are equal to each other, and therefore:

$$\begin{aligned}A =&2\left(\int_0^{\frac{\pi}{3}} 2\sin(x)-\tan(x)dx\right)\\ =&2\left[-2\cos(x)-\ln(\sec(x))\right]_0^{\frac{\pi}{3}}\\ =&2\left(-2\frac{1}{2}-\ln(2)+2-\ln(1)\right)\\ =&2\left(-1-\ln(2)+2-0\right)\\ =&2\left(1-\ln(2)\right)\\ =&2-2\ln(2)\end{aligned}$$

6.1.42. $\int_{-\frac{1}{2}}^{\frac{1}{2}} 1-|y|-2y^2dy = \int_{-\frac{1}{2}}^{0} 1+y-2y^2dy + \int_0^{\frac{1}{2}} 1-y-2y^2dy = -\frac{7}{24}+\frac{7}{24} = \boxed{\frac{7}{6}}$.
(to find the points of intersection, solve $2y^2 = 1-|y|$, and split up into the two cases $y\geq 0$ and $y<0$). Also, it might help to notice that your function is even, so you really only care about the case where $y\geq 0$.

6.1.43. Here $n=5$, and $D \approxeq 2(f(1)+f(3)+f(5)+f(7)+f(9)) = \frac{2}{60}(2+6+9+11+12) = \boxed{117\frac{1}{3}}$, where $f(x)=v_K-v_C$ (notice that $v_K\geq v_C$ throughout the race!)

6.1.51. The first region has area equal to $\int_0^b 2\sqrt{y}dy = \frac{4}{3}b^{\frac{3}{2}}$ (notice that we're integrating with respect to y, and $y=x^2 \Leftrightarrow y=\pm\sqrt{x}$. Also, draw a picture to see why we have an extra factor of 2 in the integral). The second region has area equal to $\int_b^4 2\sqrt{y}dy = -\frac{4}{3}b^{\frac{3}{2}}+\frac{32}{3}$, so to solve for b, we need to set those two areas equal:

$$\frac{4}{3}b^{\frac{3}{2}} = -\frac{4}{3}b^{\frac{3}{2}}+\frac{32}{3} \Leftrightarrow \frac{8}{3}b^{\frac{3}{2}} = \frac{32}{3} \Leftrightarrow b^{\frac{3}{2}}=4 \Leftrightarrow b=4^{\frac{2}{3}}$$

Section 6.2: Volumes

Note: In case you're confused by what I mean with K, Outer, Inner, etc., make sure to check out the 'Volumes'-Handout on my website.

6.2.6. Disk method, $K = 0$, $x = e^y$, so $\int_1^2 \pi(e^y)^2 dy = \int_1^2 \pi(e^{2y})dy = \boxed{\frac{\pi}{2}(e^4 - e^2)}$

6.2.13. Washer method, $K = 1$, Outer $= (3) - 1 = 2$, Inner $= (1 + \sec(x)) - 1 = \sec(x)$, Points of intersection $\pm\frac{\pi}{3}$, so:

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \pi(2^2 - \sec^2(x))dx = \pi(4\frac{2\pi}{3} - \tan(\frac{\pi}{3}) + \tan(\frac{-\pi}{3})) = \pi(\frac{8\pi}{3} - 2\sqrt{3}) = 2\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)$$

6.2.17. Washer method, $K = -1$, and notice $y = x^2 \Leftrightarrow x = \sqrt{y}$ (in this case $x \geq 0$), Outer $= \sqrt{y} - (-1) = \sqrt{y} + 1$, Inner $= y^2 - (-1) = y^2 + 1$, Point of intersection $y = 0$ and $y = 1$, so:

$$\int_0^1 \pi(\sqrt{y} + 1)^2 - (y^2 + 1)^2 dy = \frac{29\pi}{30}$$

6.2.47. Disk method, $K = 0$, $\int_0^h \pi\left(r - \frac{r}{h}x\right)^2 dx = \boxed{\frac{\pi}{3}r^2 h}$ (the point is to rotate the usual cone by $90°$ so that its height lies on the $x-$axis, and the base disk lies on the $y-$axis., and this it's easy to use the disk method!)

6.2.49. Disk method, $K = 0$, $\int_{r-h}^r \pi(\sqrt{r^2 - y^2})^2 dy = \int_{r-h}^r \pi(r^2 - y^2)dy \boxed{\pi h^2\left(r - \frac{1}{3}h\right)}$ (use the fact that $x^2 + y^2 = r^2$, and solve for y)

6.2.55. $A(x) = \frac{1}{2}L^2 = \frac{1}{2}(\frac{b}{\sqrt{2}})^2 = \frac{1}{4}b^2 = \frac{1}{4}(2y)^2 = y^2 = \frac{36-9x^2}{4} = 9 - \frac{9}{4}x^2$ (here L is the length of a side of the triangle, and $b = 2y$ is the hypotenuse) so $V = \int_{-2}^2 \left(9 - \frac{9}{4}x^2\right) dx = \boxed{24}$ (you get the endpoints by setting $y = 0$ in $9x^2 + 4y^2 = 36$)

6.2.65. The point is to draw a very good picture! Make one sphere have center $(0, -\frac{r}{2})$ in the $xy-$plane and the other one have center $(0, \frac{r}{2})$. Then the volume is really the volume of two pieces of equal volume, let's focus on $x \geq 0$ only! Then, using the disk method, you get:

$$V = 2\int_0^{\frac{r}{2}} \pi\left(\sqrt{r^2 - \left(x + \frac{r}{2}\right)^2}\right)^2 dx = 2\pi\int_0^{\frac{r}{2}} r^2 - \left(x + \frac{r}{2}\right) dx = \frac{5\pi r^3}{12}$$

(here we used the fact that $(x + \frac{r}{2})^2 + y^2 = r^2$, and solved for y. This looks a bit strange, but remember that your height is really on the left sphere, not on the right one!)

6.2.68. This is **much** easier with the shell method of section 6.3. Here $K = 0$, $f(x) = \sqrt{R^2 - x^2}$ (since $x^2 + y^2 = R^2$), and so $\int_r^R 2\pi x\sqrt{R^2 - x^2}dx = \boxed{\frac{2\pi}{3}\left(R^2 - r^2\right)^{\frac{3}{2}}}$ (use the substitution $u = R^2 - x^2$)

Section 6.3: Volumes by cylindrical shells

6.3.2. $\int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)dx = \boxed{2\pi}$ (use the substitution $u = x^2$)

6.3.13. Shell method: $K = 0$, $|y - 0| = y$, Outer $= 2$, Inner $= 1+(y-2)^2$, Points of intersection $y = 1$, $y = 3$, so $\int_1^3 2\pi y(2-(1+(y-2)^2))dy = \int_1^3 2\pi y(1-(y-2)^2))dy = \boxed{\frac{16\pi}{3}}$.

6.3.15. Shell method: $K = 2$, $|x - 2| = 2 - x$, Outer $= x^4$, Inner $= 0$, $\int_0^1 2\pi(2 - x)(x^4)dx = \boxed{\frac{7\pi}{15}}$

6.3.19. Shell method: $K = 1$, $|y - 1| = 1 - y$, Outer $= 1$, Inner $= \sqrt[3]{y}$, $\int_0^1 2\pi(1 - y)(1 - \sqrt[3]{y})dy = \boxed{\frac{5\pi}{14}}$

6.3.46. Shell method: $K = 0$, $|x| = x$, Outer $= \sqrt{r^2 - (x - R)^2}$ (use the fact that $(x-R)^2+y^2 = r^2$), Innter $= -\sqrt{r^2 - (x - R)^2}$, so $\int_{R-r}^{R+r} 2\pi x 2\sqrt{r^2 - (x - R)^2}dx = \boxed{\pi^2 Rr^2}$ (use the substitution $u = x - R$, and remember what you did in 5.5.73)

6.3.48. Shell method: $K = 0$, $|x| = x$, Outer $= 2\sqrt{R^2 - x^2}$ (use the fact that $x^2 + y^2 = R^2$), Inner $= 0$,

$$\int_r^R 2\pi x(2\sqrt{R^2 - x^2})dx = \frac{4\pi}{3}(R^2 - r^2)^{\frac{3}{2}} = \frac{4\pi}{3}\left(\left(\frac{h}{2}\right)^2\right)^{\frac{3}{2}} = \frac{4\pi}{3}\frac{h^3}{8} = \frac{\pi h^3}{6}$$

(use the substitution $u = R^2 - x^2$, and the fact that $r^2 + (\frac{h}{2})^2 = R^2$ by the Pythagorean theorem)